

INVARIANT DIFFERENTIAL OPERATORS ON A CLASS OF MULTIPLICITY FREE SPACES

HUBERT RUBENTHALER

ABSTRACT. If Q is a non degenerate quadratic form on \mathbb{C}^n , it is well known that the differential operators $X = Q(x)$, $Y = Q(\partial)$, and $H = E + \frac{n}{2}$, where E is the Euler operator, generate a Lie algebra isomorphic to \mathfrak{sl}_2 . Therefore the associative algebra they generate is a quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$. This fact is in some sense the foundation of the metaplectic representation. The present paper is devoted to a generalization where $Q(x)$ is replaced by $\Delta_0(x)$ which is a relative invariant of a multiplicity free representation (G, V) with a one dimensional quotient (see definition below). Over these spaces we study various algebras of differential operators. In particular if $G' = [G, G]$ is the derived group of the reductive group G , we prove that the algebra $D(V)^{G'}$ of G' -invariant differential operators with polynomial coefficients on V , is a quotient of a Smith algebra over its center. Over \mathbb{C} this class of algebras was introduced by S.P. Smith [Sm] as a class of algebras similar to $\mathcal{U}(\mathfrak{sl}_2)$. This allows us to describe by generators and relations the structure of $D(V)^{G'}$. As a corollary we obtain that various "algebras of radial components" are quotients of ordinary Smith algebras over \mathbb{C} . We also give the complete classification of the multiplicity free spaces (G, V) with a one dimensional quotient, and pay particular attention to the subclass of prehomogeneous vector spaces of commutative parabolic type, for which further results are obtained.

1. INTRODUCTION

1.1. Let H be a reductive algebraic group over \mathbb{C} and let X be a smooth irreducible H -variety. Let $\mathbb{C}[X]$ be the algebra of regular functions on X and let $D(X)$ be the algebra of differential operators on X . Then the H -action on X extends naturally to $\mathbb{C}[X]$ and $D(X)$. Let $\mathbb{C}[X]^H$ (resp. $D(X)^H$) be the subalgebras of H -invariants in $\mathbb{C}[X]$ (resp. $D(X)$). The ring $\mathbb{C}[X]^H$ is the ring of regular functions on the categorical quotient $X//H$. The problem of determining the structure of $D(X)^H$ was investigated by several authors ([Sch], [VdBe], [L-S]). On the other hand under the above mentioned hypothesis there exists a H -equivariant restriction map

$$\delta : D(X)^H \longrightarrow D(X//H).$$

obtained by applying elements in $D(X)^H$ to functions in $\mathbb{C}[X]^H$. It is expected that $D(X)^H$ as well as its image under δ (the so-called algebra of radial components) should share many properties of enveloping algebras ([Sch], [Lev]). In this paper we obtain the precise structure of $D(V)^{G'}$ in the case

where (G, V) is a so called *multiplicity free spaces with one dimensional quotient*, (here G is reductive and $G' = [G, G]$ is the derived group). To be more precise we show that $D(V)^{G'}$ is a quotient of a generalized *Smith algebra* over its center which is a polynomial algebra. Over \mathbb{C} this kind of algebras were introduced by S. P. Smith ([Sm]) as natural generalizations of the enveloping algebra of \mathfrak{sl}_2 . He also shows in the same paper that these algebras have a very interesting representation theory.

1.2. Let us now give a more precise description of our paper.

In section 2 we first give the basic definitions, notations and properties of multiplicity free spaces, and more specifically of the multiplicity free spaces with one dimensional quotient. It is worthwhile noticing that a multiplicity free space is always a prehomogeneous space. We prove that the invariant differential operators on the open orbit have always polynomial coefficients (see Theorem 2.2.4). We give the classification (without proof) of the multiplicity free spaces with a one dimensional quotient (Theorem 2.3.6 and Tables 2 and 3 at the end of the paper). We also point out an important subclass of the class of multiplicity free spaces with one dimensional quotient, namely the set of parabolic prehomogeneous spaces of commutative type.

In section 3 we introduce the various algebras of differential operators we are interested in. We define their natural gradings and we define the so-called Bernstein-Sato polynomial of an homogeneous operator of any degree, not only for degree zero operators as usual. We obtain there the first results concerning these algebras. We also prove that these algebras are Noetherian and we compute their Gelfand-Kirillov dimension. Then, using the Harish-Chandra isomorphism for multiplicity free spaces which is due to F. Knop, we prove a key lemma on invariant polynomials under the so-called little Weyl group which enables us to prove that $D(V)^G$ is a polynomial algebra over the center $\mathcal{Z}(\mathcal{T})$ of $D(V)^{G'}$, with the Euler operator as generator (Theorem 3.3.6). We also give generators of the center $\mathcal{Z}(\mathcal{T})$ (Theorem 3.3.9) and obtain some specific results in the case of PV 's of commutative parabolic type. Finally we study the ideals of the algebras $D(V)^{G'}$ and $D(\Omega)^{G'}$, where Ω is the open G -orbit in V .

In section 4 we show that our algebras of differential operators can be embedded into the Weyl algebra of a one dimensional torus, but with polynomial coefficients. More precisely they are sub-algebras of $\mathbb{C}[X_1, \dots, X_r] \otimes \mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ where $r + 1$ is the rank of (G, V) .

Section 5 is devoted to the structure of $D(V)^{G'}$. We first briefly define and study the Smith algebras over a commutative ring \mathbf{A} with unit and no zero divisors (the original definition by Smith was over \mathbb{C}). These algebras are defined by generators and relations (involving an polynomial in $\mathbf{A}[t]$), and their center is a polynomial algebra $\mathbf{A}[\Omega_1]$, where Ω_1 is a generalized Casimir element. Then we prove that $D(V)^{G'}$ is isomorphic to the quotient of a Smith algebra over its center $\mathcal{Z}(\mathcal{T})$ by the two-sided ideal generated by the element Ω_1 (see Theorem 5.2.2). Concretely, we give generators and relations for $D(V)^{G'}$.

Section 6 is devoted to the study of the algebras of radial components. By radial component of a differential operator in $D(V)^{G'}$ we mean the restriction of D to a G' -isotypic component of $\mathbb{C}[V]$. As a corollary of the preceding results we prove that these algebras are quotients of "classical" Smith algebras, that is Smith algebras over \mathbb{C} (see Theorem 6.2.2). Of course the defining relations depend on the G' -isotypic component. We also give generators of the kernel of the radial component map. In the case of the trivial representation of G' , the structure of the algebra of radial components was first obtained by [Lev], by other methods.

The results of this paper were announced in [Ru-5].

Acknowledgment: I would like to thank Thierry Levasseur for providing me with the manuscript of [Lev]. I would also like to thank Sylvain Rubenthaler who provided me with a first proof of Proposition 5.1.9 which was important for my understanding.

2. MULTIPLICITY FREE SPACES WITH ONE DIMENSIONAL QUOTIENT

2.1. Prehomogeneous Vector Spaces. Basic definitions and properties.

Let G be a connected algebraic group over \mathbb{C} , and let (G, ρ, V) be a rational representation of G on the (finite dimensional) vector space V . Then the triplet (G, ρ, V) is called a *Prehomogeneous vector space* (abbreviated to *PV*) if the action of G on V has a Zariski open orbit $\Omega \in V$. For the general theory of *PV*'s, we refer the reader to the book of Kimura [Ki] or to [S-K]. The elements in Ω are called *generic*. The *PV* is said to be *irreducible* if the corresponding representation is irreducible. The *singular set* S of (G, ρ, V) is defined by $S = V \setminus \Omega$. Elements in S are called *singular*. If no confusion can arise we often simply denote the *PV* by (G, V) . We will also write $g.x$ instead of $\rho(g)x$, for $g \in G$ and $x \in V$. It is easy to see that the condition for a rational representation (G, ρ, V) to be a *PV* is in fact an infinitesimal condition. More precisely let \mathfrak{g} be the Lie algebra of G and let $d\rho$ be the derived representation of ρ . Then (G, ρ, V) is a *PV* if and only if there exists $v \in V$ such that the map:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & V \\ X & \longmapsto & d\rho(X)v \end{array}$$

is surjective (we will often write $X.v$ instead of $d\rho(X)v$). Therefore we will call (\mathfrak{g}, V) a *PV* if the preceding condition is satisfied.

Let (G, V) be a *PV*. A rational function f on V is called a *relative invariant* of (G, V) if there exists a rational character χ of G such that $f(g.x) = \chi(g)P(x)$ for $g \in G$ and $x \in V$. From the existence of an open orbit it is easy to see that a character χ which is trivial on the isotropy subgroup of an element $x \in \Omega$ determines a unique relative invariant P . Let S_1, S_2, \dots, S_k denote the irreducible components of codimension one of the singular set S . Then there exist irreducible polynomials P_1, P_2, \dots, P_k such that $S_i = \{x \in V \mid P_i(x) = 0\}$. The P_i 's are unique up to nonzero constants. It can be proved that the P_i 's are relative invariants of (G, V) and any nonzero

relative invariant f can be written in a unique way $f = cP_1^{n_1}P_2^{n_2}\dots P_k^{n_k}$, where $n_i \in \mathbb{Z}$ and $c \in \mathbb{C}^*$. The polynomials P_1, P_2, \dots, P_k are called the *fundamental relative invariants* of (G, V) . Moreover if the representation (G, V) is irreducible then there exists at most one irreducible polynomial which is relatively invariant.

The Prehomogeneous vector space (G, V) is called *regular* if there exists a relative invariant polynomial P whose Hessian $H_P(x)$ is nonzero on Ω . If G is reductive, then (G, V) is regular if and only if the singular set S is a hypersurface, or if and only if the isotropy subgroup of a generic point is reductive. If the PV (G, V) is regular, then the contragredient representation (G, V^*) is again a PV .

2.2. Multiplicity free spaces.

For the results concerning multiplicity free spaces we refer the reader to the survey by Benson and Ratcliff ([Be-Ra-1]) or to [Kn-2]. Let (G, V) be a finite dimensional rational representation of a connected reductive algebraic group G . Let $\mathbb{C}[V]$ be the algebra of polynomials on V . Then G acts on $\mathbb{C}[V]$ by

$$g \cdot \varphi(x) = \varphi(g^{-1}x) \quad (g \in G, \varphi \in \mathbb{C}[V]).$$

As the space $\mathbb{C}[V]^n$ of homogeneous polynomials of degree n is stable under this action, the representation $(G, \mathbb{C}[V])$ is completely reducible. Let $D(V)$ be the algebra of differential operators with polynomial coefficients. The group G acts also on $D(V)$ by

$$(g \cdot D)(\varphi) = g \cdot (D(g^{-1} \cdot \varphi)) \quad (g \in G, D \in D(V), \varphi \in \mathbb{C}[V]).$$

Recall the G -equivariant identifications between $\mathbb{C}[V]$ and the symmetric algebra $S(V^*)$ of the dual space V^* and between $\mathbb{C}[V^*]$ and the symmetric algebra $S(V)$ of V . The embedding
$$\begin{array}{ccc} V & \longrightarrow & D(V) \\ v & \longmapsto & D_v \end{array}$$
 where $D_v P(x) =$

$\lim_{t \rightarrow 0} \frac{P(x+tv) - P(x)}{t}$ extends uniquely to an embedding $S(V) \longrightarrow D(V)$ whose image is the ring of differential operators with constant coefficients. If $f \in S(V) \simeq \mathbb{C}[V^*]$ we denote by $f(\partial)$ the corresponding differential operator. Another way to construct $f(\partial)$ for $f \in \mathbb{C}[V^*]$ is to say that $f(\partial)$ is the unique differential operator on V satisfying

$$f(\partial_x) e^{\langle x, y \rangle} = f(y) e^{\langle x, y \rangle} \quad (x \in V, y \in V^*) \quad (2-2-1)$$

Recall also that the $\mathbb{C}[V]$ -module $D(V)$ can be identified with $\mathbb{C}[V] \otimes S(V)$ through the multiplication map

$$\begin{array}{ccc} m : \mathbb{C}[V] \otimes S(V) & \xrightarrow{\simeq} & D(V) \\ \varphi \otimes f & \longmapsto & \varphi f(\partial) \end{array}$$

The preceding map is in fact G -equivariant and therefore the G -module $D(V)$ is isomorphic to the G -module $\mathbb{C}[V] \otimes S(V)$. The duality pairing $V \otimes V^* \longrightarrow \mathbb{C}$ extends uniquely to the non-degenerate G -equivariant pairing

$$\begin{array}{ccc} S(V) \otimes S(V^*) \simeq \mathbb{C}[V^*] \otimes \mathbb{C}[V] & \longrightarrow & \mathbb{C} \\ f \otimes \varphi & \longmapsto & \langle f, \varphi \rangle = f(\partial)\varphi(0) \end{array} \quad (2-2-2)$$

which gives rise to an embedding $\mathbb{C}[V^*] \hookrightarrow \mathbb{C}[V]^*$. It is easy to see that $\langle \mathbb{C}^i[V^*], \mathbb{C}^j[V] \rangle = \{0\}$ if $i \neq j$

Definition 2.2.1. *Let G be a connected reductive algebraic group, and let V be the space of a finite dimensional (complex) rational representation of G . The representation (G, V) is said to be multiplicity free if each irreducible representation of G occurs at most once in the representation $(G, \mathbb{C}[V])$.*

Let us give some results concerning MF (= multiplicity free) spaces (see [Be-Ra-1], [H-U], [Kn-2]):

Theorem 2.2.2.

- 1) *A finite dimensional representation (G, V) is MF if and only if (B, V) is a prehomogeneous vector space for any Borel subgroup B of G (and hence each MF space (G, V) is a PV).*
- 2) *A finite dimensional representation (G, V) is MF if and only if the algebra $D(V)^G$ of invariant differential operators with polynomial coefficients is commutative.*
- 3) *If (G, V) is a MF space, then the dual space (G, V^*) is also MF .*

Proof. The first assertion is due to Vinberg and Kimelfeld ([V-K]), another proof can be found in [Kn-2]. The second assertion is due to Howe and Umeda ([H-U], Theorem 7.1). For the third assertion note that as $\langle \mathbb{C}^i[V^*], \mathbb{C}^j[V] \rangle = \{0\}$, we obtain that $f \mapsto \langle f, \cdot \rangle$ is a G -equivariant isomorphism between $\mathbb{C}^i[V^*]$ and $\mathbb{C}^i[V]^*$, and hence (G, V^*) is multiplicity free. \square

Note that the commutativity of $D(V)^G$ for a MF space is just a consequence of the definition, since we have a simultaneous diagonalization of all the operators in $D(V)^G$.

Let us be more precise about the decompositions of the polynomials under the action of the group G or a Borel subgroup. Therefore we need more notations. We can write $G = G'C$ where $G' = [G, G]$ is the subgroup of commutators, and where $C = Z(G)^\circ \simeq (\mathbb{C}^*)^p$ is the connected component of the center of G . Let T' be a maximal torus in G' , and let $B' = T'U$ be a Borel subgroup of G' , where U is the nilradical of B' . The group $T = T'C$ is a maximal torus in G and $B = TU$ is a Borel subgroup of G . We will denote by $\mathfrak{g}, \mathfrak{g}', \mathfrak{t}, \mathfrak{t}', \mathfrak{c}, \mathfrak{b}, \mathfrak{b}', \mathfrak{u}$ the corresponding Lie algebras. Let R be the set of roots of $(\mathfrak{g}', \mathfrak{t}')$, let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be the basis of simple roots corresponding to \mathfrak{b}' and let R^+ be the corresponding set of positive roots. Denote by Λ' the lattice of weights of $(\mathfrak{g}', \mathfrak{t}')$. We have $\Lambda' = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \dots \oplus \mathbb{Z}\omega_\ell$ where the ω_i 's are the fundamental weights. Let $\Lambda'^+ = \mathbb{N}\omega_1 \oplus \mathbb{N}\omega_2 \oplus \dots \oplus \mathbb{N}\omega_\ell$ be the set of dominant weights. Denote by $X(C)$ the group of algebraic characters of C , which we will sometimes consider as linear forms on \mathfrak{c} . Set:

$$\Lambda = \Lambda' \oplus X(C), \quad \Lambda^+ = \Lambda'^+ \oplus X(C).$$

For $\lambda \in \Lambda^+$ (resp. $\lambda' \in \Lambda'^+$) let us denote by $V_{-\lambda}$ (resp. $V_{-\lambda'}$) an irreducible \mathfrak{g} -module (resp. \mathfrak{g}' -module) with highest weight λ (resp. λ'). We use this unusual notation because we want to index the modules occurring in $\mathbb{C}[V]$ by the character of their highest weight polynomials, rather than by the highest weight.

For a multiplicity free space (G, V) we have the decomposition:

$$\mathbb{C}[V] = \bigoplus_{\lambda \in \Lambda^+} V_{-\lambda}^{m(\lambda)}$$

where $m(\lambda) = 0$ or 1 . If $m(\lambda) = 1$, then there exists a uniquely defined positive integer $d(\lambda)$ such that $V_{-\lambda} \in \mathbb{C}[V]^{d(\lambda)}$. The integer $d(\lambda)$ is called the *degree* of λ . Let us denote by $\Delta_0, \Delta_1, \dots, \Delta_k, \dots, \Delta_r$ the fundamental relative invariants invariants of the PV (B, V) , indexed in such a way that $\Delta_0, \Delta_1, \dots, \Delta_k$ are the fundamental relative invariants of the PV (G, V) and such that Δ_r is the highest weight vector of V^* which must automatically appear as it is of degree 1. Then any relative invariant of (B, V) is of the form $c\Delta^{\mathbf{a}}$ where $\mathbf{a} = (a_0, a_1, \dots, a_r) \in \mathbb{Z}^{r+1}$ and where $\Delta^{\mathbf{a}} = \Delta_0^{a_0} \dots \Delta_r^{a_r}$. The non negative integer $r+1$ is called the *rank* of the MF space (G, V) . The algebra of U -invariants is the subalgebra generated by the Δ_i 's, i.e. $\mathbb{C}[V]^U = \mathbb{C}[\Delta_0, \dots, \Delta_r]$. It is worthwhile noticing that, as the polynomials Δ_i are algebraically independent from the general theory of PV 's, this latter algebra is a polynomial algebra. Let λ_i be the character of Δ_i (we use the same notation λ_i for the character of the group and for its derivative, which is an element of Λ^+). Hence the (infinitesimal) character of $\Delta^{\mathbf{a}}$ is $\lambda_{\mathbf{a}} = a_0\lambda_0 + \dots + a_r\lambda_r$. Of course by definition the elements $\Delta^{\mathbf{a}}$ ($a_i \geq 0, i = 0, \dots, r$) are the highest weights vectors in $\mathbb{C}[V]$. Due to the fact that the group action on $\Delta^{\mathbf{a}}$ is given by $g.\Delta^{\mathbf{a}}(x) = \Delta^{\mathbf{a}}(g^{-1}x)$, the infinitesimal highest weight of $\Delta^{\mathbf{a}}$ is $-\lambda_{\mathbf{a}} = -a_0\lambda_0 - \dots - a_r\lambda_r$.

If we set $V_{\mathbf{a}} = V_{-\lambda_{\mathbf{a}}}$, we therefore can write

$$\mathbb{C}[V] = \bigoplus_{a_0 \geq 0, \dots, a_r \geq 0} V_{\mathbf{a}} \quad (2-2-3)$$

Sometimes, if $\lambda = a_0\lambda_0 + \dots + a_r\lambda_r$, we simply write V_{λ} instead of $V_{\mathbf{a}}$. If we denote by d_i the degree of Δ_i , one can notice that all elements in $V_{\mathbf{a}}$ are of degree $d(\mathbf{a}) = a_0d_0 + a_1d_1 + \dots + a_rd_r$. It is also worthwhile noticing that, as an easy consequence of the multiplicity free decomposition of $\mathbb{C}[V]$, we have

$$V_{\mathbf{a}} = \Delta_0^{a_0} \Delta_1^{a_1} \dots \Delta_k^{a_k} V_{0, \dots, 0, a_{k+1}, \dots, a_r} \quad (2-2-4)$$

As far as we know the following useful lemma has never been noticed.

Lemma 2.2.3.

Define $\Omega = \{x \in V \mid \Delta_i(x) \neq 0, i = 0, \dots, k\}$. Let $\mathbb{C}[\Omega]$ be the ring of regular functions on Ω (elements of $\mathbb{C}[\Omega]$ are just rational functions whose denominators are of the form $\Delta_0^{a_0} \dots \Delta_k^{a_k}$, with $a_0, \dots, a_k \geq 0$). As the polynomials $\Delta_0, \dots, \Delta_k$ are relative invariants under G , the open set Ω is G -stable, and therefore G acts on $\mathbb{C}[\Omega]$. Then $\mathbb{C}[\Omega]$ decomposes without multiplicities under the action of G . More precisely the decomposition into irreducibles is given by

$$\mathbb{C}[\Omega] = \bigoplus_{\substack{(a_0, \dots, a_k) \in \mathbb{Z}^{k+1} \\ (a_{k+1}, \dots, a_r) \in \mathbb{N}^{r-k}}} V_{\mathbf{a}}$$

where $V_{\mathbf{a}} = \Delta_0^{a_0} \Delta_1^{a_1} \dots \Delta_k^{a_k} V_{0, \dots, 0, a_{k+1}, \dots, a_r}$ is the irreducible subspace of $\mathbb{C}[\Omega]$ generated by the highest weight vector $\Delta^{\mathbf{a}} = \Delta_0^{a_0} \Delta_1^{a_1} \dots \Delta_r^{a_r}$.

Proof. It is clear that the spaces $V_{\mathbf{a}}$ in the decomposition are G -irreducible. As they are generated by the highest weight vectors $\Delta^{\mathbf{a}}$ which have distinct weights, the sum is direct. Conversely let $f \in \mathbb{C}[\Omega]$. Without lack of generality we can suppose that $f = \frac{P}{\Delta_0^{a_0} \dots \Delta_k^{a_k}}$, with $a_0, \dots, a_k \geq 0$, and with $P \in V_{\mathbf{b}}$ where $\mathbf{b} \in \mathbb{N}^{r+1}$. Hence $f \in V_{\mathbf{b}'}$ where $\mathbf{b}' = \mathbf{b} - (a_0, \dots, a_k, 0, \dots, 0) \in \mathbb{Z}^{k+1} \times \mathbb{N}^{r-k}$. \square

This lemma has the following consequence.

Theorem 2.2.4.

Let (G, V) be a multiplicity free space. As before set $\Omega = \{x \in V \mid \Delta_i(x) \neq 0, i = 0, \dots, k\}$. Then $D(V)^G = D(\Omega)^G$. In other words any G -invariant differential operator with coefficients in $\mathbb{C}[\Omega]$ has in fact polynomial coefficients.

Proof. Let $D \in D(\Omega)^G$. As we know from the preceding Lemma that $\mathbb{C}[\Omega]$ decomposes without multiplicities under G , we obtain that D defines a G -equivariant endomorphism on each $V_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^{k+1} \times \mathbb{N}^{r-k}$. Therefore D stabilizes $\mathbb{C}[V] = \bigoplus_{a_0 \geq 0, \dots, a_r \geq 0} V_{\mathbf{a}}$. It is easy to see that a differential operator with rational coefficients and which stabilizes the polynomials must have polynomial coefficients. \square

Remark 2.2.5. The preceding theorem applies in the particular case of Jordan algebras, or equivalently (through the so-called Kantor-Koecher-Tits construction) in the case of PV 's of commutative type (see section 2.4 below). Let V be a simple Jordan algebra over \mathbb{C} or \mathbb{R} . Let Ω be the set of invertible elements in V and let G be the structure group of V . It can be proved that (G, V) is a multiplicity free space and then our results implies that $D(V)^G = D(\Omega)^G$. This result is usually obtained by computing an explicit set of generators (see [No], [Y] or [F-K]).

Proposition 2.2.6.

Let (G, V) be a MF space. For $(a_{k+1}, \dots, a_r) \in \mathbb{N}^{r-k}$ set $\tilde{\mathbf{a}} = (0, \dots, 0, a_{k+1}, \dots, a_r) \in \mathbb{N}^{r+1}$. The spaces $V_{\mathbf{a}} = \Delta_0^{a_0} \dots \Delta_k^{a_k} V_{\tilde{\mathbf{a}}}$ are G' -equivalent if $\tilde{\mathbf{a}}$ is fixed and $(a_0, \dots, a_k) \in \mathbb{Z}^{k+1}$. If we define

$$U_{\tilde{\mathbf{a}}} = \bigoplus_{(a_0, \dots, a_k) \in \mathbb{N}^{k+1}} \Delta_0^{a_0} \dots \Delta_k^{a_k} V_{\tilde{\mathbf{a}}}, \quad W_{\tilde{\mathbf{a}}} = \bigoplus_{(a_0, \dots, a_k) \in \mathbb{Z}^{k+1}} \Delta_0^{a_0} \dots \Delta_k^{a_k} V_{\tilde{\mathbf{a}}},$$

then the decomposition of $\mathbb{C}[V]$ and $\mathbb{C}[\Omega]$ in G' -isotypic components are given by

$$\mathbb{C}[V] = \bigoplus_{\tilde{\mathbf{a}}} U_{\tilde{\mathbf{a}}}, \quad \mathbb{C}[\Omega] = \bigoplus_{\tilde{\mathbf{a}}} W_{\tilde{\mathbf{a}}}$$

Proof. The map $P \mapsto \Delta_0^{a_0} \dots \Delta_k^{a_k} P$ is a G' -equivariant isomorphism between $V_{\tilde{\mathbf{a}}}$ and $\Delta_0^{a_0} \dots \Delta_k^{a_k} V_{\tilde{\mathbf{a}}}$, hence all these spaces are G' -equivalent. To prove the second assertion it is enough to prove that if $\tilde{\mathbf{a}} \neq \tilde{\mathbf{b}}$, then the spaces $V_{\tilde{\mathbf{a}}}$ and $V_{\tilde{\mathbf{b}}}$ are not G' -equivalent. Suppose that this would be the case and let $\Delta^{\tilde{\mathbf{a}}}$ and $\Delta^{\tilde{\mathbf{b}}}$ be the corresponding highest weight vectors with characters $\lambda_{\tilde{\mathbf{a}}}$ and $\lambda_{\tilde{\mathbf{b}}}$ respectively. From the G' -equivalence we know that $\lambda_{\tilde{\mathbf{a}}}|_{\mathfrak{u}'} = \lambda_{\tilde{\mathbf{b}}}|_{\mathfrak{u}'}$.

and hence $P = \frac{\Delta_{\tilde{\mathbf{a}}}}{\Delta_{\tilde{\mathbf{b}}}}$ is a relative invariant under B whose character is trivial on \mathfrak{t}' . Therefore it generates a one dimensional representation, hence P is a relative invariant under G . Finally we obtain that $\Delta_{\tilde{\mathbf{a}}} = \Delta_0^{a_0} \dots \Delta_k^{a_k} \Delta_{\tilde{\mathbf{b}}}$, and this is not possible if $\tilde{\mathbf{a}} \neq \tilde{\mathbf{b}}$. \square

As (G, V^*) is multiplicity free (Theorem 2.2.2) and as we have remarked that $\mathbb{C}^i[V^*] \simeq \mathbb{C}^i[V]^*$, we have

$$\mathbb{C}[V^*] = \bigoplus_{a_0 \geq 0, \dots, a_r \geq 0} V_{\mathbf{a}}^* \quad (2-2-5)$$

where $V_{\mathbf{a}}^*$ is the irreducible G -submodule of $\mathbb{C}[V^*]$ generated by a lowest weight vector $\Delta^{*\mathbf{a}} \in \mathbb{C}[V^*]$, defined up to a multiplicative constant, whose character with respect to the opposite Borel subgroup B^- is equal to $-\lambda_{\mathbf{a}} = -a_0\lambda_0 - \dots - a_r\lambda_r$. Let us fix a lowest weight vector Δ_i^* ($i = 0, \dots, r$) with character $-\lambda_i$ (with respect to B^-). Then we can choose $\Delta^{*\mathbf{a}} = \Delta_0^{a_0} \Delta_1^{a_1} \dots \Delta_r^{a_r}$. Of course the module $V_{\mathbf{a}}^*$ is the dual module of $V_{\mathbf{a}}$ through $f \rightarrow \langle f, \rangle$ (see (2-2-2)).

As $V_{\mathbf{a}}$ is a G -irreducible module, it is well known that the tensor G -module $V_{\mathbf{a}} \otimes V_{\mathbf{a}}^*$ contains up to constant, a unique G -invariant vector $R_{\mathbf{a}}$ and that $V_{\mathbf{a}} \otimes V_{\mathbf{b}}^*$ does not contain any non trivial G -invariant vector if $\mathbf{a} \neq \mathbf{b}$ (see for example [H-U]). To be more precise we define $R_{\mathbf{a}}$ to be the operator corresponding to the "unit matrix" in $V_{\mathbf{a}} \otimes V_{\mathbf{a}}^* \simeq \text{Hom}(V_{\mathbf{a}}, V_{\mathbf{a}})$. Moreover as $\mathbb{C}[V] \otimes \mathbb{C}[V]^*$ is G -isomorphic to $D(V)$, the element $R_{\mathbf{a}}$ can be viewed as a G -invariant differential operator with polynomial coefficients. The operators $R_{\mathbf{a}}$ are sometimes called *Capelli operators*. They are also called *unnormalized canonical invariants* in [Be-Ra-1]. Moreover the family of elements $R_{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{N}^{r+1}$) is a vector basis of the vector space $D(V)^G = D(\Omega)^G$.

The Capelli operators R_i corresponding to the space V_{λ_i} ($i = 0, \dots, r$) will be of particular importance because of the result below. It is worthwhile noticing that, as we have chosen Δ_r to be the highest weight vector of V^* , we have $R_r = E$ where E is the Euler operator.

Theorem 2.2.7. (Howe-Umeda)

Let (G, V) be a MF space. The Capelli operators R_i ($i = 0, \dots, r$) are algebraically independent and $D(V)^G = \mathbb{C}[R_0, \dots, R_r]$.

Proof. See [H-U] (Theorem 9.1) or [Be-Ra-1] (Corollary 7.4.4). \square

Remark 2.2.8. Recall that for $i = 0, 1, \dots, k$ the polynomials $\Delta_0, \Delta_1, \dots, \Delta_k$ are the fundamental relative invariants under the action of the full group G . Once these polynomials are fixed, let us define the polynomials $\Delta_i^* \in \mathbb{C}[V^*]$ as the unique fundamental relative invariant of (G, V^*) under B^- with character λ_i^{-1} , such that $\Delta_i^*(\partial)\Delta_i(0) = 1$, for $i = 0, \dots, k$. Then the Capelli operators R_i ($i = 0, \dots, k$) are given by $R_i = \Delta_i(x)\Delta_i^*(\partial)$, and the Capelli operator corresponding to the irreducible component $V_{a_0\lambda_0 + \dots + a_k\lambda_k}$ is scalar multiple of $\Delta_0^{a_0}(x) \dots \Delta_k^{a_k}(x)\Delta_0^*(\partial)^{a_0} \dots \Delta_k^*(\partial)^{a_k}$. More generally the Capelli operator $R_{\mathbf{a}}$ corresponding to $V_{\mathbf{a}}$ where $\mathbf{a} = a_0\lambda_0 + \dots + a_k\lambda_k + \dots + a_r\lambda_r$ is a scalar multiple of $\Delta_0^{a_0}(x) \dots \Delta_k^{a_k}(x)\Delta_0^*(\partial)^{a_0} \dots \Delta_k^*(\partial)^{a_k} R_{a_{k+1}\lambda_{k+1} + \dots + a_r\lambda_r}$.

2.3. Multiplicity free spaces with one dimensional quotient.

Let us now define the main objects this paper deals with.

Definition 2.3.1. (T. Levasseur, [Lev] sections 3.2 and 4.2)

1) A prehomogeneous vector space (G, V) is said to be of rank one* if there exists an homogeneous polynomial Δ_0 on V such that $\Delta_0 \notin \mathbb{C}[V]^G$ and such that $\mathbb{C}[V]^{G'} = \mathbb{C}[\Delta_0]$.

2) A multiplicity free space (G, V) is said to have a one-dimensional quotient if it is a PV of rank one.

The proof of the following Proposition will take place in a forthcoming paper on the classification of MF spaces with a one dimensional quotient.

Proposition 2.3.2.

If (G, V) is a PV of rank one, then the polynomial Δ_0 is the unique fundamental relative invariant of (G, V) . More precisely a PV (G, V) is of rank one if and only if it has a unique fundamental relative invariant.

Example 2.3.3. We will see in section 2.4 below that the regular PV's of commutative parabolic type (see Table 1) are MF spaces, with a unique fundamental invariant. Hence by Proposition 2.3.2, these PV's are examples of MF spaces with one dimensional quotients.

Let us now explain the classification of MF spaces with a one dimensional quotient.

Definition 2.3.4. (see [Kn-2])

1) Two representations (G_1, ρ_1, V_1) and (G_2, ρ_2, V_2) are called geometrically equivalent if there is an isomorphism $\Phi : V_1 \longrightarrow V_2$ such that $\Phi(\rho_1(G_1))\Phi^{-1} = \rho_2(G_2)$.

2) A MF space (G, V) is called decomposable if it is geometrically equivalent to a representation of the form $(G_1 \times G_2, V_1 \oplus V_2)$, where (G_1, V_1) and (G_2, V_2) are non-zero MF spaces. It is called indecomposable if it is not decomposable.

3) A representation (G, V) is called saturated if the dimension of the center of $\rho(G)$ is equal to the number of irreducible summands of V .

Remark 2.3.5. The notion of geometric equivalence is quite natural, once one has remarked that the notion of MF space depends only on $\rho(G)$. It is worthwhile noticing that any representation is geometrically equivalent to its dual representation. Any representation can be made saturated by adding a torus.

Historically the classification of MF spaces goes as follows. Kac ([Ka]) determined all the MF spaces where the representation (G, V) is irreducible. Brion ([Br]) did the case where $G' = [G, G]$ is (almost) simple. Finally Benson-Ratcliff and Leahy classified independently, up to geometric equivalence, all the indecomposable saturated MF -spaces (see [Be-Ra-1], [Be-Ra-2], [Lea], [Kn-2])

The proof of the following classification Theorem will appear in a forthcoming paper.

*It must be remarked that if (G, V) is also multiplicity free, then the rank as a PV is not at all the same as the rank as a MF space.

Theorem 2.3.6.

Up to geometric equivalence the complete list of indecomposable saturated MF spaces with a one dimensional quotient is given by Table 2 and Table 3 at the end of the paper.

2.4. Prehomogeneous Vector Spaces of commutative parabolic type.

The aim of this section is to describe an important subclass of the class of MF-spaces with a one dimensional quotient.

The PV's of parabolic type were introduced by the author in [Ru-1] and then developed in his thesis (1982, [Ru-2]). A convenient reference is the book [Ru-3]. The papers [M-R-S] and [R-S-1] contain also parts of the results summarized here. Sato and Kimura ([S-K], [Ki]) gave a complete classification of irreducible regular and so called *reduced PV's* with a reductive group G (*reduced* stands for a specific representative in a certain equivalence class, the so-called "castling class", the details are not needed here). It turns out that most of these PV's are of parabolic type. The class of PV's we are interested in, is a subclass of the full class of parabolic PV's, the so called PV's of commutative parabolic type. Let us now give a brief account of the results for these PV's which turn out to be MF-spaces with a one dimensional quotient.

Let $\tilde{\mathfrak{g}}$ be a simple Lie algebra over \mathbb{C} satisfying the following two assumptions:

a) There exists a decomposition $\tilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+$ which is also a 3-grading:

$$[\mathfrak{g}, V^+] \subset V^+, \quad [\mathfrak{g}, V^-] \subset V^-, \quad [V^-, V^+] \subset \mathfrak{g}$$

$$[V^+, V^+] = \{0\}, \quad [V^-, V^-] = \{0\}.$$

b) There exists a semi-simple element $H_0 \in \mathfrak{g}$ and $X_0 \in V^+, Y_0 \in V^-$ such that (Y_0, H_0, X_0) is an \mathfrak{sl}_2 -triple.

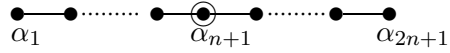
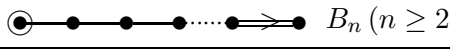
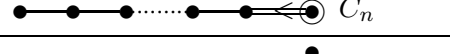
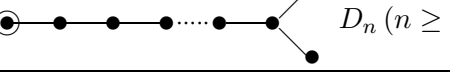

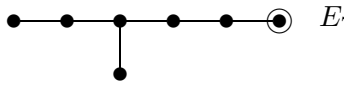
One can prove, that under the assumption a) that (\mathfrak{g}, V^+) is an irreducible PV (here the action of \mathfrak{g} on V^+ is the Lie bracket). In fact, as we will sketch now, $\mathfrak{g} \oplus V^+$ is a maximal parabolic subalgebra of $\tilde{\mathfrak{g}}$ whose nilradical V^+ is commutative, this is the reason why these PV's are called of *commutative parabolic type*. Assumption b) is equivalent to the regularity of the PV (\mathfrak{g}, V^+) . We will now describe these PV's in terms of roots. Let \mathfrak{t} be a Cartan subalgebra of \mathfrak{g} which contains H_0 . It is easy to see that \mathfrak{t} is also a Cartan subalgebra of $\tilde{\mathfrak{g}}$. Let \tilde{R} be the set of roots of the pair $(\tilde{\mathfrak{g}}, \mathfrak{t})$. The set P of roots occurring in $\mathfrak{g} \oplus V^+$ is a parabolic subset. Therefore there exists a set of simple roots $\tilde{\Psi}$, such that if \tilde{R}^+ is the corresponding set of positive roots, then $P \setminus (P \cap -P) \subset \tilde{R}^+$. Let ω be the highest root in \tilde{R} and let R be the set of roots of the pair $(\mathfrak{g}, \mathfrak{t})$. Then $\Psi = \tilde{\Psi} \cap R$ is a set of simple roots for R and $\tilde{\Psi} = \Psi \cup \{\alpha_0\}$, where α_0 has coefficient 1 in ω . What we have done up to now can be performed for all 3-gradings of $\tilde{\mathfrak{g}}$ (in other words for all decompositions of $\tilde{\mathfrak{g}}$ satisfying assumption a)). It is easy to see that the element H_0 in assumption b) can be described as the unique element in \mathfrak{t}

such that

$$\begin{cases} \alpha(H_0) = 0 & \forall \alpha \in \Psi \\ \alpha_0(H_0) = 2 \end{cases}$$

Assumption b) means just that H_0 is the semi-simple element of an \mathfrak{sl}_2 -triple. Let w_0 be the unique element of the Weyl group of \tilde{R} such that $w_0(\tilde{\Psi}) = -\tilde{\Psi}$. One can show that the preceding condition on H_0 is equivalent to the condition $w_0(\alpha_0) = -\alpha_0$. This leads to an easy classification of the regular PV 's of commutative parabolic type. From the preceding discussion we deduce that these objects are in one to one correspondence with connected Dynkin diagrams where we have circled a root α_0 , which has coefficient 1 in the highest root and such that $w_0(\alpha_0) = -\alpha_0$. In the following table we give the list of these objects and also the corresponding Lie algebra \mathfrak{g} , and the space V^+ ($MS_n(\mathbb{C})$ stands for the symmetric matrices of size n , and $AS_{2n}(\mathbb{C})$ stands for the anti-symmetric matrices of size $2n$).

Table 1

	$\tilde{\mathfrak{g}}$	\mathfrak{g}	V^+
A_{2n+1}		$\mathfrak{sl}_{n+1} \times \mathfrak{gl}_{n+1}$	$M_n(\mathbb{C})$
B_n		$\mathfrak{so}_{2n-2} \times \mathbb{C}$	\mathbb{C}^{2n-2}
C_n		\mathfrak{gl}_n	$MS_n(\mathbb{C})$
D_n^1		$\mathfrak{so}_{2n-1} \times \mathbb{C}$	\mathbb{C}^{2n-1}
D_{2n}^2		$\mathfrak{gl}_{2n}(\mathbb{C})$	$AS_{2n}(\mathbb{C})$
E_7		$E_6 \times \mathbb{C}$	\mathbb{C}^{27}

We need to get more inside the structure of the regular PV 's of commutative parabolic type. First let us define the rank of such a PV . Let \tilde{R}_1 be the set of roots which are orthogonal to α_0 (this is also the set of roots which are strongly orthogonal to α_0). The set \tilde{R}_1 is again a root system as well as $R_1 = \tilde{R}_1 \cap R$. Define

$$\mathfrak{t}_1 = \sum_{\alpha \in \tilde{R}_1} \mathbb{C}H_\alpha, \quad \tilde{\mathfrak{g}}_1 = \mathfrak{t}_1 \oplus \sum_{\alpha \in \tilde{R}_1} \tilde{\mathfrak{g}}^\alpha.$$

Then $\tilde{\mathfrak{g}}_1$ is a semi-simple Lie algebra, \mathfrak{t}_1 is a Cartan subalgebra of $\tilde{\mathfrak{g}}_1$ and the set roots of $(\tilde{\mathfrak{g}}_1, \mathfrak{t}_1)$ is \tilde{R}_1 . Moreover if we set $\mathfrak{g}_1 = \tilde{\mathfrak{g}}_1 \cap \mathfrak{g}$, $V_1^+ = \tilde{\mathfrak{g}}_1 \cap V^+$ and $V_1^- = \tilde{\mathfrak{g}}_1 \cap V^-$, then \mathfrak{t}_1 is also a Cartan subalgebra of \mathfrak{g}_1 and we have

$$\tilde{\mathfrak{g}}_1 = V_1^- \oplus \mathfrak{g}_1 \oplus V_1^+.$$

The key remark is that if we start with $(\tilde{\mathfrak{g}}, \mathfrak{g}, V^+)$ which satisfies assumption a), the preceding decomposition of \mathfrak{g}_1 is again a 3-grading satisfying assumptions a), in other words (\mathfrak{g}_1, V_1^+) is again a *PV* of commutative parabolic type (except that the algebra $\tilde{\mathfrak{g}}_1$ may be semi-simple, not necessarily simple). Moreover if $(\tilde{\mathfrak{g}}, \mathfrak{g}, V^+)$ satisfies also b), then the same is true for $(\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+)$.

Let α_1 be the root which plays the role of α_0 for the new *PV* of commutative parabolic type $(\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+)$. We can apply the same procedure, called the *descent*, to $(\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+)$ and so on and then we will obtain inductively a sequence

$$\cdots \subset (\tilde{\mathfrak{g}}_k, \mathfrak{g}_k, V_k^+) \subset \cdots \subset (\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+) \subset (\tilde{\mathfrak{g}}, \mathfrak{g}, V^+)$$

of *PV*'s of commutative parabolic type. This sequence stops because, for dimension reasons, there exists an integer r such that $\tilde{R}_r \neq \emptyset$ and such that $\tilde{R}_{r+1} = \emptyset$.

The integer $r + 1$ is then called the *rank* of (\mathfrak{g}, V^+) . We will see below that *PV*'s of commutative parabolic type are always *MF* spaces with a one dimensional quotient. The preceding defined rank turns out to be same as the rank as a *MF* space which was defined in section 2.2.

Let us denote by $\alpha_0, \alpha_1, \dots, \alpha_r$ the set of strongly orthogonal roots occurring in V^+ which appear in the descent (the rank is also the number of elements in this sequence, it can be characterized as the maximal number of strongly orthogonal roots occurring in V^+). One proves also that if the first *PV* (\mathfrak{g}, V^+) is regular (*i.e.* if it satisfies b)), then the same is true for all spaces (\mathfrak{g}_i, V_i^+) ($i = 0, \dots, r$) and $V_r^+ = \mathbb{C}X_{\alpha_r}$, where $X_{\alpha_r} \in \tilde{\mathfrak{g}}$, $X_{\alpha_r} \neq 0$.

Let \tilde{G} be the adjoint group of the $\tilde{\mathfrak{g}}$ and let G be the analytic subgroup of \tilde{G} corresponding to \mathfrak{g} . The group G is also the centralizer of H_0 in \tilde{G} . As we have already noticed, the representation (G, Ad, V^+) is then an irreducible *PV*. Let G_r be the subgroup of G corresponding to \mathfrak{g}_r . The descent process described before leads to a sequence of *PV*'s:

$$(G_r, V_r^+) \subset (G_{r-1}, V_{r-1}^+) \subset \cdots \subset (G_1, V_1^+) \subset (G, V^+) \quad (2-4-1)$$

The orbital structure of (G, V^+) can be described as follows. Let us denote as usual by X_γ a non zero element of $\tilde{\mathfrak{g}}^\gamma$.

Define

$$\begin{aligned} I_0^+ &= X_{\alpha_0}, I_1^+ = X_{\alpha_0} + X_{\alpha_1}, \dots, I_k^+ = X_{\alpha_0} + X_{\alpha_1} + \cdots + X_{\alpha_k}, \dots, \\ I_r^+ &= I^+ = X_{\alpha_0} + X_{\alpha_1} + \cdots + X_{\alpha_r}. \end{aligned}$$

Then the set

$$\{0, I_0^+, I_1^+, \dots, I_r^+ = I^+\}$$

is a set of representatives of the G -orbits in V^+ (there are $\text{rank}(G, V^+) + 1 = r + 2$ orbits). The orbit $G \cdot I^+$ is the open orbit $\Omega^+ \subset V^+$.

The Killing form \tilde{B} of $\tilde{\mathfrak{g}}$ allows us to identify V^- with the dual space of V^+ and the representation (G, V^-) becomes then the dual *PV* of (G, V^+) . One can similarly perform a descent on the V^- side, and obtain a sequence

$$(G_r, V_r^-) \subset (G_{r-1}, V_{r-1}^-) \subset \cdots \subset (G_1, V_1^-) \subset (G, V^-) \quad (2-4-2)$$

of PV 's, where the groups are the same as in $(2-4-1)$. The PV (G_i, V_i^-) is dual to (G_i, V_i^+) . The set of elements

$$\{0, I_0^-, I_1^-, \dots, I_r^- = I^-\}$$

where $I_i^- = X_{-\alpha_0} + X_{-\alpha_1} + \dots + X_{-\alpha_i}$ is a set of representatives of the G -orbits in V^- . The orbit $G.I^-$ is the open orbit $\Omega^- \subset V^-$. We will always choose the elements $X_{-\alpha_i}$ such that $(X_{-\alpha_i}, H_{\alpha_i}, X_{\alpha_i})$ is a \mathfrak{sl}_2 -triple. If the PV (\mathfrak{g}, V^+) satisfies the assumptions a) and b), then $H_0 = H_{\alpha_0} + H_{\alpha_1} + \dots + H_{\alpha_n}$ and (I^-, H_0, I^+) is a \mathfrak{sl}_2 -triple. More generally, under the same hypothesis, if $H_i = H_{\alpha_0} + H_{\alpha_1} + \dots + H_{\alpha_i}$, then (I_i^-, H_i, I_i^+) is a \mathfrak{sl}_2 -triple.

We suppose from now on that (G, V^+) is regular. Remember that this means that it satisfies assumption b). Let Δ_0 be the unique irreducible polynomial on V^+ which is relatively invariant under the action of G . Let Δ_1 be the unique irreducible polynomial on V_1^+ which is relatively invariant under the group G_1 . We have $V^+ = W_1^+ \oplus V_1^+$ where W_1^+ is the sum of the root spaces of V^+ which do not occur in V_1^+ . Therefore for $x = y_1 + x_1$ ($x_1 \in V_1^+, y_1 \in W_1^+$), we can define $\Delta_1(x) = \Delta_1(x_1)$ and hence the polynomial Δ_1 can be viewed as a polynomial on V^+ . Inductively we can define a sequence $\Delta_0, \Delta_1, \dots, \Delta_r$ of irreducible polynomials on V^+ , where the polynomial Δ_i depends only on the variables in V_i^+ and is, in general, not relatively invariant under G , but under $G_i \subset G$. It can be shown that $\partial^\circ(\Delta_i) = r + 1 - i = \text{rank}(G, V^+) - i$.

Let $H \subset G$ be the isotropy subgroup of I^+ and let $\mathfrak{h} \subset \mathfrak{g}$ be its Lie algebra. Another striking fact concerning the irreducible regular PV 's of commutative parabolic type is that the open orbit $\Omega^+ \simeq G/H$ is a symmetric space. This means that \mathfrak{h} is the fixed points set of an involution of \mathfrak{g} (this involution can be shown to be $\exp \text{ad}(I^+) \exp \text{ad}(I^-) \exp \text{ad}(I^+)$). Let us denote by B^- the Borel subgroup of G defined by $-\Psi$.

One can show that V^+ is already a PV under the action of B^- . In other words (G, V^+) is a MF space (See [M-R-S], Theorem 3.6 p.110). More precisely the fundamental relative invariants of (B^-, V^+) are the polynomials $\Delta_0, \Delta_1, \dots, \Delta_r$. The open B^- -orbit in V^+ is the set

$$\mathcal{O}^+ = \{x \in V^+ \mid \Delta_0(x)\Delta_1(x)\dots\Delta_r(x) \neq 0\}.$$

In fact, as (G, V) is irreducible, the polynomial Δ_0 is the unique fundamental relative invariant under G and hence (G, V) is a MF space with a one dimensional quotient.

Symmetrically, if B^+ is the Borel subgroup of G defined by Ψ , the representation (B^+, V^-) is also a PV , and the fundamental relatively invariant polynomials is a set

$$\{\Delta_0^*, \Delta_1^*, \dots, \Delta_r^*\}$$

of irreducible polynomials where $\partial^\circ(\Delta_i^*) = r + 1 - i = \text{rank}(G, V^+) - i$. Of course these polynomial are obtained by a descent process similar to the one described before on V^+ . Moreover the polynomial Δ_0^* is the fundamental relatively invariant of (G, V^-) .

Let \tilde{B} be the Killing form on $\tilde{\mathfrak{g}}$. Then for any polynomial P^* on V^- , we define a differential operator $P^*(\partial)$ with constant coefficients on V^+ by the

formula

$$P^*(\partial)e^{\tilde{B}(x,y)} = P^*(y)e^{\tilde{B}(x,y)} \text{ for } x \in V^+, y \in V^-. \quad (2-4-3)$$

Remark 2.4.1. Let G/K be a hermitian symmetric space, let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^+$ be the usual decomposition of the complexified Lie algebra of G . Then of course the preceding decomposition is a 3-grading of $\mathfrak{g}_{\mathbb{C}}$. Moreover this 3-grading verifies assumption b) at the beginning of section 2.2. if and only if G/K is of tube type. Conversely it can be shown that any PV of commutative parabolic type can be obtained this way from a hermitian symmetric space of tube type.

Similarly if J is a simple Jordan algebra over \mathbb{C} , and if $struc(J)$ is its structure algebra, then from the Kantor-Koecher-Tits construction ([Ko], [Ti]) it is known that one can put a Lie algebra structure on $J \oplus struc(J) \oplus J$, and this decomposition is a 3-grading which verifies assumption b). Conversely if (G, V^+) is a PV of commutative parabolic type, then one can define on V^+ a Jordan product which makes V^+ into a simple Jordan algebra (see for example [F-K]).

3. ALGEBRAS OF DIFFERENTIAL OPERATORS

In this section we suppose that (G, V) is a MF space with a one dimensional quotient, and we will use the notations introduced in the precedent section, especially in subsections 2.3 and 2.4.

3.1. Gradings and Bernstein-Sato polynomials.

Recall that we denote by $\Delta_0, \Delta_1, \dots, \Delta_r$ the fundamental relative invariants under a fixed Borel subgroup B of G . As the space has a one dimensional quotient, Δ_0 is the unique polynomial among them which is relatively invariant under G (this means that $k = 0$ in the notations of section 2.2). We also set $\Omega = \{x \in V \mid \Delta_0(x) \neq 0\}$.

The Euler operator E on V is defined for $P \in \mathbb{C}[V]$ by

$$EP(x) = \frac{\partial}{\partial t} P(tx)_{t=1} = P'(x)x.$$

Remember from section 2.2 that we have chosen Δ_r in such a way that $R_r = E$.

Once and for all we also define the following two elements in $D(V)$:

$$X = \Delta_0 \text{ (multiplication by } \Delta_0), \quad Y = \Delta_0^*(\partial).$$

The operator

$$X^{-1} \text{ (multiplication by } \Delta_0^{-1})$$

which belongs to $D(\Omega)$ will also play an important role. From the definition of the G action on $\mathbb{C}[V]$ and on $D(V)$ we have

$$g.X = \lambda_0(g^{-1})X, \quad g.X^{-1} = \lambda_0(g)X^{-1}, \quad g.Y = \lambda_0(g)Y \quad (3-1-1)$$

and hence $X, Y \in D(V)^{G'}$ and $X^{-1} \in D(\Omega)^{G'}$.

Let us now introduce the following notations that we will use in the rest of the paper:

$$\mathcal{T} = D(\Omega)^{G'}, \quad \mathcal{T}_0 = D(V)^G = D(\Omega)^G$$

(the last equality comes from Theorem 2.2.4). Remember that \mathcal{T}_0 is a polynomial algebra in $r + 1$ variables (Theorem 2.2.7). We have the following inclusions:

$$\mathcal{T}_0 = D(V)^G = D(\Omega)^G \subset D(V)^{G'} \subset \mathcal{T} = D(\Omega)^{G'}.$$

As differential operators in \mathcal{T} have coefficients which are fractions whose denominators are homogeneous (powers of Δ_0), it is clear that \mathcal{T} is graded by its homogeneous components (an element D in \mathcal{T} is said to be of degree m if $[E, D] = mD$). But on the other hand any homogeneous element D in \mathcal{T} preserves the G' -isotypic components $W_{\mathbf{a}} = \bigoplus_{n \in \mathbb{N}} \Delta_0^n V_{\mathbf{a}}$ (see Proposition 2.2.6). Therefore an homogeneous element D maps $\Delta_0^n V_{\mathbf{a}}$ on $\Delta_0^{n+j} V_{\mathbf{a}}$ for some j and hence only multiples of d_0 (the degree of Δ_0) occur as homogeneous degrees in \mathcal{T} . If we define, for $p \in \mathbb{Z}$, $\mathcal{T}_p = \{D \in \mathcal{T} \mid [E, D] = pd_0 D\}$, then

$$\mathcal{T} = \bigoplus_{p \in \mathbb{Z}} \mathcal{T}_p \quad (3-1-2)$$

(At this point it is not completely evident that the two definitions of \mathcal{T}_0 coincide, that is that $D(V)^G = \{D \in \mathcal{T} \mid [E, D] = 0\}$. This will be a consequence of the proof of Proposition 3.1.7 below).

Similarly if we define

$$D(V)_p^{G'} = \{D \in D(V)^{G'} \mid [E, D] = pd_0 D\},$$

we have $D(V)^{G'} = \bigoplus_{p \in \mathbb{Z}} D(V)_p^{G'}$.

Definition 3.1.1. For $\mathbf{a} = (a_0, a_1, \dots, a_r)$ and $p \in \mathbb{N}$, we define $\mathbf{a} + p = (a_0 + p, a_1, \dots, a_r)$. Then if $D \in \mathcal{T}_p$, the Schur Lemma ensures that if $P \in V_{\mathbf{a}}$ we have $DP = b_D(\mathbf{a}) \Delta_0^p P$ where $b_D(\mathbf{a}) \in \mathbb{C}$. It is easy to see that b_D is a polynomial in the variables (a_0, a_1, \dots, a_r) (see for example [Kn-2], proof of Corollary 4.4). This polynomial is called the Bernstein-Sato polynomial of D .

Example 3.1.2. Relations $(3-1-1)$ imply that $X \in \mathcal{T}_1$, $X^{-1} \in \mathcal{T}_{-1}$ and $Y \in \mathcal{T}_{-1}$. And of course $E \in \mathcal{T}_0$. Obviously, from the definition, we have $b_X(\mathbf{a}) = b_{X^{-1}}(\mathbf{a}) = 1$, $b_E(\mathbf{a}) = d_0 a_0 + d_1 a_1 + \dots + d_r a_r =$ the degree of $V_{\mathbf{a}}$ (recall that d_i is the degree of Δ_i). From $(3-1-1)$ we obtain that $Y \in \mathcal{T}_{-1}$. The computation of b_Y is more difficult. However it is known in the case of PV 's of commutative parabolic type. In this case, for $\mathbf{X} = (X_0, X_1, \dots, X_r)$ it is given by

$$b_Y(\mathbf{X}) = c \prod_{j=0}^r (X_0 + \dots + X_j + j \frac{d}{2}) \quad (3-1-3)$$

where the constant c can be made explicit (see [B-R], Théorème 3.19) and where $\frac{d}{2} = \frac{\dim(V) - d_0}{(d_0 - 1)d_0}$. This explicit computation of the polynomial b_Y in the particular case of PV 's of commutative parabolic type has been obtained by several authors, using distinct methods (see [B-R], [F-K], [K-S], [Wa]). It is worthwhile noticing that the constant d is the same as the constant d which is familiar to specialists of Jordan algebras (see Remark 2.4.1).

Remark 3.1.3. It is known from Igusa ([I]) that for any homogeneous polynomial P of degree > 2 , the Lie algebra generated over \mathbb{C} by the two

differential operators $P(x)$ and $P(\partial)$ is infinite dimensional. However in the particular case of regular PV 's of commutative parabolic type and for $X = \Delta_0(x)$ and $Y = \Delta_0(\partial)$ an easy proof of this fact can be obtained by using the explicit knowledge of b_Y given before (see [Ru-4], Théorème 3.1.). If Δ_0 is a non degenerate quadratic form (this corresponds also to a commutative parabolic PV), then the Lie algebra generated over \mathbb{C} by X and Y is well known to be isomorphic to \mathfrak{sl}_2 . This remark is the foundation of the so-called metaplectic (or Segal-Shale-Weil) representation ([H],[Sh],[We]).

The following Lemma is obvious, but useful.

Lemma 3.1.4.

Let $D_1, D_2 \in \mathcal{T}_p$. Then $D_1 = D_2$ if and only if $b_{D_1} = b_{D_2}$.

Definition 3.1.5. The automorphism τ of $\mathcal{T} = D(\Omega)^{G'}$ is defined by

$$\forall D \in \mathcal{T}, \tau(D) = XDX^{-1}$$

Proposition 3.1.6.

The algebra \mathcal{T}_0 is stable under τ and for any $D \in \mathcal{T}_0$ we have

$$XD = \tau(D)X \quad (3-1-4)$$

$$DY = Y\tau(D) \quad (3-1-5)$$

Proof. By definition $\mathcal{T}_0 = D(V)^G = D(\Omega)^G$. From relations (3-1-1) we see that if D is G -invariant so is $\tau(D)$. Obviously $\tau(D) \in D(\Omega)^G$. Hence \mathcal{T}_0 is τ -stable. Relation (3-1-4) is just the definition of τ . We will now prove that (3-1-5) holds on each subspace $V_{\mathbf{a}}$. Let b_D be the Bernstein-Sato polynomial of D . Then an easy calculation shows that the left and right side of (3-1-5) act on $V_{\mathbf{a}}$ by $b_D(\mathbf{a}-1)b_Y(\mathbf{a})X^{-1}$. Then Lemma 3.1.4 implies (3-1-5). \square

Let us denote by $\mathcal{T}_0[X, Y]$ the subalgebra of \mathcal{T} generated by \mathcal{T}_0 , X and Y . From the preceding Proposition and from the fact that XY and YX belong to \mathcal{T}_0 we know that any element $D \in \mathcal{T}_0[X, Y]$ can be written as a finite sum $D = \sum_{p,q \in \mathbb{N}} a_{p,q} X^p Y^q$ with $a_{p,q} \in \mathcal{T}_0$. Similarly, let $\mathcal{T}_0[X, X^{-1}]$ denote the subalgebra of \mathcal{T} generated by \mathcal{T}_0 , X and X^{-1} . Any element D in $\mathcal{T}_0[X, X^{-1}]$ can be written as a finite sum $D = \sum_{p \in \mathbb{Z}} a_p X^p$. The following Proposition shows that $D(V)^{G'} = \mathcal{T}_0[X, Y]$ and that $\mathcal{T} = D(\Omega)^{G'} = \mathcal{T}_0[X, X^{-1}]$ and makes the gradings more precise.

Proposition 3.1.7.

1) We have

$$D(V)^{G'} = \mathcal{T}_0[X, Y] = (\bigoplus_{p \in \mathbb{N}^*} \mathcal{T}_0 Y^p \oplus) \oplus \mathcal{T}_0 \oplus (\bigoplus_{p \in \mathbb{N}} \mathcal{T}_0 X^p)$$

(in particular $D(V)_p^{G'} = \mathcal{T}_0 X^p$ if $p \geq 0$, and $D(V)_p^{G'} = \mathcal{T}_0 Y^{-p}$ if $p < 0$).

Equivalently we have

$$D(V)^{G'} = \mathcal{T}_0[X, Y] = (\bigoplus_{p \in \mathbb{N}^*} Y^p \mathcal{T}_0 \oplus) \oplus \mathcal{T}_0 \oplus (\bigoplus_{p \in \mathbb{N}} X^p \mathcal{T}_0).$$

2) We have $\mathcal{T} = D(\Omega)^{G'} = \mathcal{T}_0[X, X^{-1}] = \bigoplus_{p \in \mathbb{Z}} \mathcal{T}_0 X^p = \bigoplus_{p \in \mathbb{Z}} X^p \mathcal{T}_0$.

3) Any element D in $\mathcal{T}_0[X, Y]$ can be written uniquely in the form

$$D = \sum_{i > 0} u_i Y^i + \sum_{i \geq 0} v_i X^i \text{ or } D = \sum_{i > 0} Y^i u_i + \sum_{i \geq 0} X^i v_i \text{ (finite sums)}$$

with $u_i, v_i \in \mathcal{T}_0$.

Any element $D \in \mathcal{T}$ can be written uniquely in the form

$$D = \sum_{i \in \mathbb{Z}} u_i X^i \text{ or } D = \sum_{i \in \mathbb{Z}} X^i u_i \quad (\text{finite sums})$$

with $u_i \in \mathcal{T}_0$.

Proof. 1) From Proposition 2.2.6 we know that the decomposition of $\mathbb{C}[V]$ into G' -isotypic components is given by

$$\mathbb{C}[V] = \bigoplus_{\tilde{\mathbf{a}} \in \mathbb{N}^r} U_{\tilde{\mathbf{a}}} \text{ where } U_{\tilde{\mathbf{a}}} = \bigoplus_{a_0 \in \mathbb{N}} \Delta_0^{a_0} V_{\tilde{\mathbf{a}}} \text{ and } \tilde{\mathbf{a}} = (0, a_1, \dots, a_r).$$

We will now use the technique of Howe and Umeda ([H-U]) which we have already mentioned before Theorem 2.2.7. As $\mathbb{C}[V] \otimes \mathbb{C}[V]^*$ is G' -isomorphic to $D(V)$, each subspace $\Delta_0^{a_0} V_{\tilde{\mathbf{a}}} \otimes (\Delta_0^{b_0} V_{\tilde{\mathbf{a}}})^*$ will give rise to a unique G' -invariant differential operator $R_{a_0, b_0, \tilde{\mathbf{a}}}$. Then by the same arguments as in Remark 2.2.8, it is easy to see that $R_{a_0, b_0, \tilde{\mathbf{a}}} = \Delta_0(x)^{a_0} R_{0,0,\tilde{\mathbf{a}}} \Delta_0^*(\partial)^{b_0} = X^{a_0} R_{0,0,\tilde{\mathbf{a}}} Y^{b_0}$. The elements $X^{a_0} R_{0,0,\tilde{\mathbf{a}}} Y^{b_0}$ ($a_0, b_0 \in \mathbb{N}, \tilde{\mathbf{a}} \in \mathbb{N}^r$) form a vector basis of $D(V)^{G'}$. Remark now that $R_{0,0,\tilde{\mathbf{a}}}$ is in $D(V)^G = \mathcal{T}_0$. Then from Proposition 3.1.6, we get $X^{a_0} R_{0,0,\tilde{\mathbf{a}}} Y^{b_0} = \tau^{a_0}(R_{0,0,\tilde{\mathbf{a}}}) X^{a_0} Y^{b_0}$ and $\tau^{a_0}(R_{0,0,\tilde{\mathbf{a}}}) \in \mathcal{T}_0$. If now $a_0 \leq b_0$, then $X^{a_0} R_{0,0,\tilde{\mathbf{a}}} Y^{b_0} = R Y^{b_0-a_0}$, where $R = \tau^{a_0}(R_{0,0,\tilde{\mathbf{a}}}) X^{a_0} Y^{a_0} \in \mathcal{T}_0$. If $a_0 > b_0$, then $X^{a_0} R_{0,0,\tilde{\mathbf{a}}} Y^{b_0} = R X^{a_0-b_0}$ where $R = \tau^{a_0}(R_{0,0,\tilde{\mathbf{a}}}) \tau^{a_0-b_0}(X^{b_0} Y^{b_0}) \in \mathcal{T}_0$. The first decomposition in assertion 1) is proved. The second decomposition is a consequence of relations (3-1-4) and (3-1-5).

2) From (3-1-2) it is enough to prove that $\mathcal{T}_p = \mathcal{T}_0 X^p$ for all $p \in \mathbb{Z}$. If $D \in \mathcal{T}_p$, then $D = (DX^{-p})X^p$ and $DX^{-p} \in \mathcal{T}_0$.

As we have obtained $\mathcal{T} = \bigoplus_{p \in \mathbb{Z}} \mathcal{T}_p$ with $\mathcal{T}_0 = D(V)^G$, we get $\{D \in \mathcal{T} \mid [E, D] = 0\} \subset D(V)^G$, and hence that $\{D \in \mathcal{T} \mid [E, D] = 0\} = D(V)^G$. Therefore the two definitions of \mathcal{T}_0 coincide.

As $D(\Omega)$ has no zero divisors assertion 3) is a consequence of 1) and 2). \square

Remark 3.1.8. The inclusion $D(V)^{G'} \subset D(\Omega)^{G'}$ is obviously strict ($X^{-1} \in D(\Omega)^{G'} \setminus D(V)^{G'}$), but the preceding results shows that they have the same "positive part" ($\bigoplus_{p \in \mathbb{N}} \mathcal{T}_0 X^p$).

The following proposition shows that all the Bernstein-Sato polynomials are known if one knows the Bernstein-Sato polynomials of Y and of the elements of \mathcal{T}_0 .

Proposition 3.1.9.

Let $D = D_0 X^n$ ($n \in \mathbb{Z}$), resp. $D = D_0 Y^n$ ($n \in \mathbb{N}^*$), $D_0 \in \mathcal{T}_0$, be generic homogeneous elements in $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$ or $\mathcal{T}_0[X, Y]$. Then $b_D(\mathbf{a}) = b_{D_0}(\mathbf{a}+n)$, resp. $b_D(\mathbf{a}) = b_{D_0}(\mathbf{a}-n)b_Y(\mathbf{a})b_Y(\mathbf{a}-1)\dots b_Y(\mathbf{a}+n-1)$.

Proof. Consider first the case where $D = D_0 X^n$ and let $Q \in V_{\mathbf{a}}$. We have:

$$DQ = b_D(\mathbf{a})\Delta_0^n Q = D_0 \Delta_0^n Q = b_{D_0}(\mathbf{a}+n)\Delta_0^n Q.$$

Hence $b_D(\mathbf{a}) = b_{D_0}(\mathbf{a}+n)$.

Consider now the case $D = D_0 Y^n$ and let $Q \in V_{\mathbf{a}}$. We have:

$$\begin{aligned} DQ &= b_D(\mathbf{a})\Delta_0^{-n}Q = D_0 Y^n Q = D_0 Y^{n-1} Y Q = D_0 Y^{n-1} b_Y(\mathbf{a})\Delta_0^{-1}Q \\ &= D_0 b_Y(\mathbf{a})b_Y(\mathbf{a}-1)\dots b_Y(\mathbf{a}+n-1)\Delta_0^{-n}Q \\ &= b_Y(\mathbf{a}-1)\dots b_Y(\mathbf{a}+n-1)b_{D_0}(\mathbf{a}-n)\Delta_0^{-n}Q. \end{aligned}$$

Hence $b_D(\mathbf{a}) = b_Y(\mathbf{a}-1)\dots b_Y(\mathbf{a}+n-1)b_{D_0}(\mathbf{a}-n)$. □

3.2. Noetherianity and Gelfand-Kirillov dimension.

Recall that a non commutative ring \mathcal{R} is said to be noetherian if the right and the left ideals are finitely generated, or equivalently if the right and the left ideals verify the ascending chain condition (see for example [MC-R] or [G-W]).

Recall also that the rings $\mathcal{T}_0[X]$, $\mathcal{T}_0[X^{-1}]$, $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$, $\mathcal{T}_0[Y]$, $\mathcal{T}_0[X, Y]$ are defined to be the subrings of $\mathbf{D}(\Omega^+)$ generated by \mathcal{T}_0 and by the elements $X, X^{-1}, \{X, X^{-1}\}$, Y and $\{X, Y\}$ respectively.

Let S be a ring and let $\sigma \in \text{Aut}(S)$. Let us recall that a σ -derivation of S is an additive map $\delta : S \rightarrow S$ such that $\delta(st) = s\delta(t) + \delta(s)\sigma(t)$. Given a σ -derivation δ , the skew polynomial ring over S determined by σ and δ is the ring $S[T, \sigma, \delta] := \langle S, T \rangle / \{sT - T\sigma(s) - \delta(s) | s \in S\}$, where $\langle S, T \rangle$ stands for the ring freely generated by S and an element T with the relations given by the ring structure on S (for details see [MC-R], section 1.2, p.15 or [G-W] p.34). Similarly one defines the skew Laurent polynomials ring by $S[T, T^{-1}, \sigma] := \langle S, T, T^{-1} \rangle / \{sT = T\sigma(s)\}$.

Theorem 3.2.1.

The rings $\mathcal{T}_0[X]$, $\mathcal{T}_0[X^{-1}]$, $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$, $\mathcal{T}_0[Y]$, $\mathcal{T}_0[X, Y]$ are noetherian.

Proof. Recall from Proposition 3.1.6 that for all $D \in \mathcal{T}_0$ one has $XD = \tau(D)X$, $YD = \tau^{-1}(D)Y$, where $\tau(D) = XDX^{-1}$. Recall also from Proposition 3.1.7 that any element $D \in \mathcal{T} = \mathcal{T}_0[X, X^{-1}]$ can be written uniquely $D = \sum_{i \in \mathbb{Z}} u_i X^i$, with $u_i \in \mathcal{T}_0$. The same easy argument shows that any element $D \in \mathcal{T}_0[Y]$ can be written uniquely $D = \sum_{i \in \mathbb{Z}} u_i Y^i$ with $u_i \in \mathcal{T}_0$.

These remarks imply that the rings $\mathcal{T}_0[X]$, $\mathcal{T}_0[X^{-1}]$ and $\mathcal{T}_0[Y]$ (which are subrings of $\mathbf{D}(\Omega^+)$) are respectively isomorphic to the "abstract" skew polynomial rings $\mathcal{T}_0[T, \tau^{-1}, 0]$, $\mathcal{T}_0[T, \tau, 0]$ and $\mathcal{T}_0[T, \tau, 0]$. They are therefore noetherian by Theorem 1.2.9. of [MC-R].

The ring $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$ is similarly isomorphic to the skew Laurent polynomials ring $\mathcal{T}_0[T, T^{-1}, \tau^{-1}]$ and is therefore noetherian by Theorem 1.4.5. of [MC-R].

The relations $XD = \tau(D)X$, $YD = \tau^{-1}(D)Y$, where $D \in \mathcal{T}_0$, imply that $\mathcal{T}_0 X = X \mathcal{T}_0$ and $\mathcal{T}_0 Y = Y \mathcal{T}_0$. Moreover $[X, Y] \in \mathcal{T}_0$. These remarks imply that $\mathcal{T}_0[X, Y]$ is an almost normalizing extension of \mathcal{T}_0 in the sense of [MC-R] (section 1.6.10.). As \mathcal{T}_0 is noetherian, this implies by Theorem 1.6.14. of [MC-R], that $\mathcal{T}_0[X, Y]$ is noetherian. □

We will denote by $GK.\dim(\mathcal{R})$ the Gelfand-Kirillov dimension of the algebra \mathcal{R} .

Theorem 3.2.2.

One has

$$\begin{aligned} GK.\dim(\mathcal{T}) &= GK.\dim(\mathcal{T}_0[X]) = GK.\dim(\mathcal{T}_0[X^{-1}]) = GK.\dim(\mathcal{T}_0[Y]) \\ &= GK.\dim(\mathcal{T}_0[X, Y]) = GK.\dim(\mathcal{T}_0) + 1 = r + 2 \end{aligned}$$

Proof. We have seen in the proof of Theorem 3.2.1 that the algebras $\mathcal{T}_0[X]$, $\mathcal{T}_0[X^{-1}]$, $\mathcal{T}_0[Y]$ are isomorphic to the skew polynomial algebra $\mathcal{T}_0[T, \tau]$ (or $\mathcal{T}_0[T, \tau^{-1}]$) and that the algebra $\mathcal{T}_0[X, X^{-1}]$ is isomorphic to the skew Laurent polynomial ring $\mathcal{T}_0[T, T^{-1}, \tau^{-1}]$.

An automorphism ν of \mathcal{T}_0 is called locally algebraic if for any $D \in \mathcal{T}_0$, the set $\{\nu^n(D), n \in \mathbb{N}\}$ spans a finite dimensional vector space. We know from [L-M-O] (Prop.1) that if ν is locally algebraic then $GK.\dim(\mathcal{T}_0[T, \nu]) = GK.\dim(\mathcal{T}_0[T, T^{-1}, \nu]) = GK.\dim(\mathcal{T}_0) + 1$ (see also [Z]).

Let us prove that τ is locally algebraic in the preceding sense. The elements $D \in \mathcal{T}_0$ are in one to one correspondence with their Bernstein-Sato polynomial b_D . We have $b_{\tau(D)}(\mathbf{a}) = b_{XDX^{-1}}(\mathbf{a}) = b_D(\mathbf{a} - 1)$. Therefore the Bernstein-Sato polynomials $b_{\tau^n(D)}$ have the same degree as b_D . Hence the space spanned by the family $b_{\tau^n(D)}$ is finite dimensional, and τ is locally algebraic.

$$\begin{aligned} GK.\dim(\mathcal{T}_0[X]) &= GK.\dim(\mathcal{T}_0[X^{-1}]) = GK.\dim(\mathcal{T}_0[Y]) \\ &= GK.\dim(\mathcal{T}_0[X, X^{-1}]) = GK.\dim(\mathcal{T}_0) + 1. \end{aligned}$$

As $\mathcal{T}_0[X] \subset \mathcal{T}_0[X, Y] \subset \mathcal{T}_0[X, X^{-1}]$, we have also $GK.\dim(\mathcal{T}_0[X, Y]) = GK.\dim(\mathcal{T}_0) + 1$. As \mathcal{T}_0 is a polynomial algebra in $r + 1$ variables (Theorem 2.2.7), we have $GK.\dim(\mathcal{T}_0) = r + 1$. □

3.3. The Harish-Chandra isomorphism and the center of \mathcal{T} .

The aim of this subsection is to describe $\mathcal{T}_0 = D(V)^G$ as a module over the center of \mathcal{T} . For this we will use the Harish-Chandra isomorphism for MF spaces due to F. Knop.

Let (G, V) be a MF space with a one dimensional quotient. Let B be a fixed Borel subgroup of G . Remember that (B, V) is a PV . We denote by $\Delta_0, \Delta_1, \dots, \Delta_r$ the set of fundamental relative invariants of (B, V) . Then, due to the one dimensional quotient hypothesis, Δ_0 is the unique fundamental relative invariant under G . We denote by d_i (resp. λ_i) the degree (resp. the infinitesimal character) of Δ_i . Let \mathfrak{b} be the Lie algebra of B and let $\mathfrak{t} \subset \mathfrak{b}$ be a Cartan subalgebra of \mathfrak{g} and let $\Sigma(\mathfrak{g}, \mathfrak{t})$ be the corresponding set of roots. Denote by W the Weyl group of $\Sigma(\mathfrak{g}, \mathfrak{t})$. Denote by $\Sigma^+(\mathfrak{g}, \mathfrak{t})$ the set of positive roots such that $\mathfrak{b} = \mathfrak{t} + \sum_{\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}^\alpha$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{t})} \alpha$. We define

$$\mathfrak{a}^* = \oplus_{i=0}^r \mathbb{C}\lambda_i \subset \mathfrak{t}^* \text{ and } A = \mathfrak{a}^* + \rho \subset \mathfrak{t}^*.$$

Let $\mathcal{Z}(\mathfrak{g})$ be the center of the enveloping algebra of \mathfrak{g} . Denote by $\mathbb{C}[\mathfrak{t}^*]^W$ the W -invariant polynomials on \mathfrak{t}^* . One knows that the classical Harish-Chandra isomorphism is an isomorphism $H : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}[\mathfrak{t}^*]^W$ which can be computed the following way. For any $\lambda \in \mathfrak{t}^*$, let V_λ be the irreducible highest weight module with highest weight λ . It is well known that $\mathcal{Z}(\mathfrak{g})$ acts by

scalar multiplication on V_λ . The scalar by which an element $z \in \mathcal{Z}(\mathfrak{g})$ acts on V_λ is precisely $H(z)(\lambda + \rho)$.

The natural representation of G on $\mathbb{C}[V]$ extends to a representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ on the same space $\mathbb{C}[V]$. Hence $z \in \mathcal{Z}(\mathfrak{g})$ acts on $V_{\mathbf{a}}$ by the scalar $H(\lambda_{\mathbf{a}} + \rho)$ where $\lambda_{\mathbf{a}} = \sum_{i=0}^r a_i \lambda_i$ (remember that $\mathbf{a} = (a_0, \dots, a_r)$). Conversely if $\lambda = a_0 \lambda_0 + \dots + a_r \lambda_r$ we define $\mathbf{a}_\lambda = (a_0, \dots, a_r) \in \mathbb{C}^{r+1}$.

On the other hand any $D \in D(V)^G = \mathcal{T}_0$ acts on each $V_{\mathbf{a}}$ by the scalar $b_D(\mathbf{a})$, where $b_D(\mathbf{a})$ is the Bernstein-Sato polynomial of D . This allows us to define the map:

$$\begin{aligned} h : D(V)^G &\longrightarrow \mathbb{C}[A] \\ D &\longmapsto h(D) : \lambda + \rho \longmapsto h(D)(\lambda + \rho) = b_D(\mathbf{a}_\lambda) \end{aligned}$$

where $\mathbb{C}[A]$ denotes the algebra of polynomials on the affine space $A = \mathfrak{a}^* + \rho \subset \mathfrak{t}^*$.

Let $\pi(z)$ be the operator in $D(V)^G$ which represents the action of z on $\mathbb{C}[V]$ and let $r : \mathbb{C}[\mathfrak{t}^*]^W \longrightarrow \mathbb{C}[A]$ be the restriction homomorphism. It is clear from the definitions that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \xrightarrow{H} & \mathbb{C}[\mathfrak{t}^*]^W \\ \pi \downarrow & & \downarrow r \\ D(V)^G & \xrightarrow{h} & \mathbb{C}[A] \end{array}$$

Theorem 3.3.1.

(Knop, see [Kn-2] Th. 4.8 and Corollary 4.9 or [Be-Ra-1], Th. 9.2.1)

The homomorphism h is injective and there exists a finite group W_0 (sometimes called the little Weyl group) which is a subgroup of the stabilizer of A in W , such that the image of h is $\mathbb{C}[A]^{W_0}$. Hence h is an isomorphism between $D(V)^G$ and $\mathbb{C}[A]^{W_0}$. The homomorphism h is called the Harish-Chandra isomorphism for the MF space (G, V) . Moreover W_0 acts as a reflection group on \mathfrak{a}^ .*

Let us see what is the automorphism of $\mathbb{C}[A]^{W_0}$ which corresponds to the action of τ through the Harish-Chandra isomorphism h . Let $D \in D(V)^G$. Then $h(\tau(D))(\lambda + \rho) = h(XDX^{-1})(\lambda + \rho) = b_{XDX^{-1}}(\lambda) = b_D(\lambda - \lambda_0)$. This calculation proves of course that $\mathbb{C}[A]^{W_0}$ is stable under $P(\lambda + \rho) \longmapsto P((\lambda - \lambda_0) + \rho)$. Therefore we make the following definition.

Definition 3.3.2. *By abuse of notation τ will also denote the automorphism of $\mathbb{C}[A]^{W_0}$ which is defined by $\tau(P)(\lambda + \rho) = P((\lambda - \lambda_0) + \rho)$ ($P \in \mathbb{C}[A]^{W_0}$). Let $\mathbb{C}[A]^{W_0, \tau}$ denote the set of elements in $\mathbb{C}[A]^{W_0}$ which are invariant under τ .*

Proposition 3.3.3.

Let $\mathcal{Z}(\mathcal{T})$ be the center of $\mathcal{T} = D(\Omega)^{G'}$. Then $\mathcal{Z}(\mathcal{T})$ is also the center of $\mathcal{T}_0[X, Y] = D(V)^{G'}$. Moreover the following assertions are equivalent:

- i) $D \in \mathcal{Z}(\mathcal{T})$*
- ii) $D \in \mathcal{T}_0$ and $\tau(D) = D$ (i.e. D commutes with X).*

iii) $D \in \mathcal{T}_0$ and the Bernstein-Sato polynomial $b_D(a_0, a_1, \dots, a_r)$ does not depend on a_0 .

iv) $D \in \mathcal{T}_0$ and $h(D) \in \mathbb{C}[A]^{W_0, \tau}$.

Proof. i) \Rightarrow ii): Let $D \in \mathcal{Z}(\mathcal{T})$. Then $[E, D] = 0$, hence $D \in \mathcal{T}_0$, and $[D, X] = 0$.

ii) \Rightarrow iii): Let $D \in \mathcal{T}_0$. If $XD = DX$ then, from the definitions we have $b_{XD}(a_0, a_1, \dots, a_r) = b_D(a_0, a_1, \dots, a_r) = b_{DX}(a_0, a_1, \dots, a_r) = b_D(a_0 + 1, a_1, \dots, a_r)$, hence $b_D(a_0, a_1, \dots, a_r)$ does not depend on a_0 .

iii) \Rightarrow i): Suppose that for $D \in \mathcal{T}_0$, the Bernstein-Sato polynomial does not depend on a_0 . Then the elements XD and DX in \mathcal{T}_1 have the same Bernstein-Sato polynomial. Hence $XD = DX$ (Lemma 3.1.4). Then from Proposition 3.1.7 3) we see that $D \in \mathcal{Z}(\mathcal{T})$. The equivalence of iii) and iv) is obvious as $h(D)(\lambda + \rho) = b_D(\lambda)$.

From ii) we obtain that $\mathcal{Z}(\mathcal{T})$ is also the center of $\mathcal{T}_0[X, Y]$. □

Remark 3.3.4. As a consequence of the preceding Proposition it is worthwhile noticing that if $D \in \mathcal{T}$ and commutes with E and X , then D commutes with Y . This is a well known property if (X, E, Y) is an \mathfrak{sl}_2 -triple (but this is not the case except if Δ_0 is a quadratic form). We will see that the associative algebra generated by X, E, Y over $\mathcal{Z}(\mathcal{T})$ is "similar" to $\mathcal{U}(\mathfrak{sl}_2(\mathcal{Z}(\mathcal{T})))$, see Theorem 5.2.2 below.

Remember that d_i denotes the degree of Δ_i for $i = 0, \dots, r$. Define a linear form μ on \mathfrak{a}^* by

$$\mu(a_0\lambda_0 + \dots + a_r\lambda_r) = \sum_{i=0}^r a_i d_i = b_E(\mathbf{a}) \quad (\mathbf{a} = (a_0, \dots, a_r) \in \mathbb{C}^{r+1})$$

(μ is the *degree form*, as its values coincide with the degree of the polynomials in $V_{\mathbf{a}}$ when $\mathbf{a} = (a_0, \dots, a_r) \in \mathbb{N}^{r+1}$).

Define also

$$\mathcal{M} = \{\lambda \in \mathfrak{a}^* \mid \mu(\lambda) = 0\} \text{ and } M = \mathcal{M} + \rho \subset A.$$

Note that $M = \{\lambda + \rho \in A \mid h(E)(\lambda + \rho) = 0\}$. As $h(E)$ is W_0 -invariant, so is the set M . Set

$$I(M) = \{P \in \mathbb{C}[A]^{W_0} \mid P|_M = 0\}.$$

The key lemma is the following.

Lemma 3.3.5.

We have $I(M) = \mathbb{C}[A]^{W_0} h(E)$ and

$$\mathbb{C}[A]^{W_0} = \mathbb{C}[A]^{W_0, \tau} \oplus I(M).$$

Proof. Let $P \in I(M)$. It is a polynomial on the affine subspace $A \subset \mathfrak{t}^*$ which vanishes on M which is the set of zeros of the irreducible polynomial $h(E)$ (it is irreducible because $h(E)(\lambda + \rho)$ is a nonzero linear form in the λ variable). Therefore $P = h(E)Q$. As P and $h(E)$ are W_0 -invariant, so is also the polynomial Q . Hence $I(M) \subset \mathbb{C}[A]^{W_0} h(E)$. As the reverse inclusion is obvious we get $I(M) = \mathbb{C}[A]^{W_0} h(E)$.

Let $F = \mathbb{C}\lambda_0 \subset \mathfrak{a}^*$. As obviously $\mathfrak{a}^* = \mathcal{M} \oplus F$, we have $A = M \oplus F$. Remember that $\mathfrak{t} = \mathfrak{c} \oplus \mathfrak{t}'$ where \mathfrak{c} is the center of \mathfrak{g} . The infinitesimal character λ_0 is a character of \mathfrak{g} , and is therefore trivial on $\mathfrak{t}' \subset \mathfrak{g}'$. As any $w_0 \in W_0$ fixes pointwise the center \mathfrak{c} of \mathfrak{g} , we see that F is pointwise fixed by W_0 .

Let $Q \in \mathbb{C}[M]^{W_0}$. Define

$$\tilde{Q}(m + f) = Q(m), \quad \text{for all } m \in M, f \in F.$$

From the preceding discussion we obtain that \tilde{Q} is W_0 -invariant, in other words $\tilde{Q} \in \mathbb{C}[A]^{W_0}$. But in fact \tilde{Q} is also τ -invariant: $\tau(\tilde{Q})(m + f) = \tilde{Q}(m + f - \lambda_0) = Q(m) = \tilde{Q}(m + f)$. Hence $\tilde{Q} \in \mathbb{C}[A]^{W_0, \tau}$, in other words any W_0 -invariant polynomial on M can be extended to a (W_0, τ) -invariant polynomial on A . This extension is in fact unique: for any τ -invariant extension $\tilde{\tilde{Q}}$ of \tilde{Q} we have $\tilde{\tilde{Q}}(m + x\lambda_0) = \tilde{\tilde{Q}}(m + (x+1)\lambda_0)$ and hence $\tilde{\tilde{Q}} = \tilde{Q}$. In fact we have proved that the restriction map:

$$\begin{array}{ccc} \mathbb{C}[A]^{W_0, \tau} & \longrightarrow & \mathbb{C}[M]^{W_0} \\ P & \longmapsto & P|_M \end{array}$$

is bijective (and therefore $\mathbb{C}[A]^{W_0, \tau} \cap I(M) = \{0\}$) and the inverse map is $Q \longmapsto \tilde{Q}$. Now for $P \in \mathbb{C}[A]^{W_0}$ we can write:

$$P = \widetilde{P|_M} + (P - \widetilde{P|_M}).$$

From the discussion above we have $\widetilde{P|_M} \in \mathbb{C}[A]^{W_0, \tau}$, and $(P - \widetilde{P|_M}) \in I(M)$. \square

Theorem 3.3.6.

- 1) $\mathcal{T}_0 = D(V)^G = \mathcal{Z}(\mathcal{T}) \oplus E\mathcal{T}_0$
- 2) Any element $H \in D(V)^G$ can be uniquely written in the form

$$H = H_0 + EH_1 + E^2H_2 \cdots + E^kH_k$$

where $H_i \in \mathcal{Z}(\mathcal{T})$, $i = 1, 2, \dots, k \in \mathbb{N}$.

Proof. Through the Harish-Chandra isomorphism h , the algebra $D(V)^G = \mathcal{T}_0$ corresponds to $\mathbb{C}[A]^{W_0}$, the algebra $\mathcal{Z}(\mathcal{T})$ corresponds to $\mathbb{C}[A]^{W_0, \tau}$ and the ideal $E\mathcal{T}_0$ corresponds to $I(M)$. Therefore the first assertion is just the pull back by h of the decomposition obtained in Lemma 3.3.5.

An element $H \in D(V)^G$ can therefore be uniquely written $H = H_0 + EH^1$, with $H_0 \in \mathcal{Z}(\mathcal{T})$, and $H^1 \in \mathcal{T}_0$. By induction we obtain a decomposition $H = H_0 + EH_1 + E^2H_2 \cdots + E^{k-1}H_{k-1} + E^kH^k$ where $H_0, \dots, H_{k-1} \in \mathcal{Z}(\mathcal{T})$, and $H^k \in \mathcal{T}_0$. The process stops because if k is greater than the degree in a_0 of b_H , then necessarily $H^k \in \mathcal{Z}(\mathcal{T})$ (see Proposition 3.3.3). \square

From the preceding Theorem and Proposition 3.1.7 we obtain immediately the following corollary.

Corollary 3.3.7.

- 1) Let $D \in \mathcal{T}$, then D can be written uniquely in the form:

$$D = \sum_{k \in \mathbb{Z}, \ell \in \mathbb{N}} H_{k, \ell} E^\ell X^k \quad \text{or} \quad D = \sum_{k \in \mathbb{Z}, \ell \in \mathbb{N}} H_{k, \ell} X^k E^\ell \quad (\text{finite sums})$$

where $H_{k,\ell} \in \mathcal{Z}(\mathcal{T})$

2) Let $D \in \mathcal{T}_0[X, Y]$, then D can be written uniquely in the form:

$$\begin{aligned} D &= \sum_{k \in \mathbb{N}^*, \ell \in \mathbb{N}} H_{k,\ell} E^\ell Y^k + \sum_{r \in \mathbb{N}, s \in \mathbb{N}} H'_{r,s} E^s X^r \text{ (finite sum) or} \\ D &= \sum_{k \in \mathbb{N}^*, \ell \in \mathbb{N}} H_{k,\ell} Y^k E^\ell + \sum_{r \in \mathbb{N}, s \in \mathbb{N}} H'_{r,s} X^r E^s \text{ (finite sum)} \end{aligned}$$

where $H_{k,\ell}, H'_{r,s} \in \mathcal{Z}(\mathcal{T})$

Corollary 3.3.8. *Let $P \in \mathbb{C}[A]^{W_0}$. Then P can be uniquely written in the form*

$$P(\lambda + \rho) = \sum_{i=0}^p \alpha_i(\lambda + \rho)(a_0 d_0 + a_1 d_1 + \cdots + a_r d_r)^i$$

where $\alpha_i \in \mathbb{C}[A]^{W_0, \tau}$ and where $\lambda = a_0 \lambda_0 + a_1 \lambda_1 + \cdots + a_r \lambda_r$.

Proof. As $h(E)(\lambda + \rho) = a_0 d_0 + a_1 d_1 + \cdots + a_r d_r$, the preceding decomposition is just the image through the Harish-Chandra isomorphism of the decomposition in Theorem 3.3.6 2). \square

Let us make some remarks about the W_0 -action on A . In fact it is easy to see that as W_0 stabilizes the affine space $A = \mathfrak{a}^* + \rho$ it also stabilizes \mathfrak{a}^* (this is implicit in Theorem 3.3.1). Moreover if we denote by 0_ρ the barycenter of the W_0 -orbit of ρ , then 0_ρ is a fixed point of the W_0 -action on A which is in M . As $\mathbb{C}[A]^{W_0} = \mathbb{C}[\mathfrak{a}^* + \rho]^{W_0} = \mathbb{C}[\mathfrak{a}^* + 0_\rho]^{W_0} \simeq \mathbb{C}[\mathfrak{a}^*]^{W_0}$, and as $\mathcal{T}_0 = D(V)^G \simeq \mathbb{C}[A]^{W_0}$ is a polynomial algebra in $r + 1$ variables by Theorem 2.2.7, the group W_0 acts as a reflection group on \mathfrak{a}^* by the Shephard-Todd-Chevalley Theorem (this is Knop's argument). Hence by the Theorem of Chevalley, the $r + 1$ algebraically independent generators of the algebra $\mathbb{C}[A]^{W_0} \simeq \mathbb{C}[\mathfrak{a}^*]^{W_0}$ can be chosen to be homogeneous, either as functions on the vector space \mathfrak{a}^* , or as functions on A , for the vector space structure on A defined by taking 0_ρ as origin. The polynomial $h(E)(\lambda + \rho) = b_E(\lambda) = a_0 \lambda_0 + \cdots + a_r \lambda_r$ is W_0 -invariant and of degree one. In general it is not homogeneous for the involved vector space structure, but nevertheless if P_0, P_1, \dots, P_r is any set of algebraically independent homogeneous generators of $\mathbb{C}[A]^{W_0}$, there must be one, say P_r which is of degree 1. Then $P_0, P_1, \dots, P_{r-1}, h(E)$ is still a set of algebraically independent generators.

We will now describe more precisely the algebra $\mathcal{Z}(\mathcal{T})$.

Theorem 3.3.9.

1) $\mathcal{Z}(\mathcal{T})$ is a polynomial algebra in r variables. For $D \in \mathcal{T}_0$, let us denote by \overline{D} the projection of D on $\mathcal{Z}(\mathcal{T})$ according to the decomposition $\mathcal{T}_0 = \mathcal{Z}(\mathcal{T}) \oplus E\mathcal{T}_0$. If $\{B_0, \dots, B_{r-1}, E\}$ is a set of algebraically independent generators of \mathcal{T}_0 , then $\{\overline{B_0}, \dots, \overline{B_{r-1}}\}$ is a set of algebraically independent generators of $\mathcal{Z}(\mathcal{T})$. In particular this is the case for the set $\{\overline{R_0}, \dots, \overline{R_{r-1}}\}$, where the R_i 's are the Capelli operators (see Theorem 2.2.7).

2) Let D be an element of \mathcal{T}_0 and let b_D be its Bernstein-Sato polynomial. Then \overline{D} is the element of $\text{End}(\mathbb{C}[V])$ which acts on each space $V_{\mathbf{a}}$ as the scalar multiplication by

$$b_{\overline{D}}(a_0, a_1, \dots, a_r) = b_D\left(-\frac{(a_1 d_1 + \cdots + a_r d_r)}{d_0}, a_1, \dots, a_r\right).$$

Proof. 1) Let us remark first that $\mathcal{Z}(\mathcal{T})$ is already known to be a polynomial algebra from a result of Knop ([Kn-1]). He has proved that for a regular action of a reductive group on a smooth affine variety the center of the ring of invariant differential operators is always a polynomial algebra. Let us give here a direct proof for the convenience of the reader. We know from Proposition 3.3.3 that $\mathcal{Z}(\mathcal{T})$ is isomorphic, through the Harish-Chandra isomorphism h , to $\mathbb{C}[A]^{W_0, \tau}$. From the proof of Lemma 3.3.5 we know that W_0 stabilizes M and that $\mathbb{C}[A]^{W_0, \tau} \simeq \mathbb{C}[M]^{W_0} = \mathbb{C}[A]^{W_0}|_M$. As W_0 is a reflection group on A (this means that it is generated by the reflections it contains), so is $W_0|_M$. Hence $\mathbb{C}[M]^{W_0}$ is a polynomial algebra in $r = \dim M$ variables by Chevalley's Theorem. If $\{B_0, \dots, B_{r-1}, E\}$ is a set algebraically independent generators of \mathcal{T}_0 , then $\{h(B_0), \dots, h(B_{r-1}), h(E)\}$ is a set of algebraically independent generators of $\mathbb{C}[A]^{W_0}$. As $h(E)|_M = 0$ we obtain that $\mathbb{C}[M]^{W_0} = \mathbb{C}[h(B_0)|_M, \dots, h(B_{r-1})|_M]$. As the transcendence degree of $\text{Frac}(\mathbb{C}[M]^{W_0})$ over \mathbb{C} is r , the generators $h(B_0)|_M, \dots, h(B_{r-1})|_M$ are algebraically independent. Taking their inverse image under h gives the first assertion of the Theorem.

2) The decomposition $\mathcal{T}_0 = \mathcal{Z}(\mathcal{T}) \oplus E\mathcal{T}_0$ is nothing else but the inverse image under h of the decomposition $\mathbb{C}[A]^{W_0} = \mathbb{C}[A]^{W_0, \tau} \oplus I(M)$. Let $\overline{D} \in \mathcal{T}_0$. From the proof of Lemma 3.3.5 we have $h(\overline{D}) = \widetilde{h(D)|_M}$, where $h(D)|_M$ is the unique (W_0, τ) -invariant extension to A of $h(D)|_M$. For $\lambda = a_0\lambda_0 + \dots + a_r\lambda_r \in \mathfrak{a}^*$, we have $h(E)(\lambda + \rho) = b_E(\lambda) = a_0d_0 + \dots + a_rd_r = \mu(\lambda)$ (the degree form). Remember also that $\mathfrak{a}^* = \mathcal{M} \oplus F$, where $F = \mathbb{C}\lambda_0$, where $\mathcal{M} = \ker(\mu)$. Let us write $\lambda = m_\lambda + \alpha\lambda_0$, according to this decomposition. Then $b_E(\lambda) = \alpha b_E(\lambda_0) = \alpha d_0$. Hence $\alpha = \frac{\mu(\lambda)}{d_0}$ and $m_\lambda = \lambda - \frac{\mu(\lambda)}{d_0}\lambda_0$. Then we obtain:

$$\begin{aligned} b_{\overline{D}}(\lambda) &= h_{\overline{D}}(\lambda + \rho) = \widetilde{h(D)|_M}(\lambda + \rho) = \widetilde{h(D)|_M}(\lambda - \frac{\mu(\lambda)}{d_0}\lambda_0 + \frac{\mu(\lambda)}{d_0}\lambda_0 + \rho) \\ &= \widetilde{h(D)|_M}(\lambda - \frac{\mu(\lambda)}{d_0}\lambda_0 + \rho) \\ &= h(D)|_M(\lambda - \frac{\mu(\lambda)}{d_0}\lambda_0 + \rho) = h(D)(\lambda - \frac{\mu(\lambda)}{d_0}\lambda_0 + \rho) \\ &= b_D(\lambda - \frac{\mu(\lambda)}{d_0}\lambda_0). \end{aligned}$$

If we translate this in the (a_0, \dots, a_r) -variables we obtain the second assertion. □

Corollary 3.3.10.

Let b_Y be the Bernstein-Sato operator of Y . For any $\ell \in \mathbb{N}$ the element of $\text{End}(\mathbb{C}[V])$ which acts on each space $V_{\mathbf{a}}$ as the scalar multiplication by

$$b_Y(-\frac{(a_1d_1 + \dots + a_rd_r)}{d_0} + \ell, a_1, \dots, a_r)$$

is the differential operator $\overline{X^{1-\ell}YX^\ell} \in \mathcal{Z}(\mathcal{T})$. Moreover, if (G, V) is a PV of commutative parabolic type, the differential operators $\overline{X^{1-\ell}YX^\ell}$ ($\ell = 0, 1, \dots, r$) are generators of $\mathcal{Z}(\mathcal{T})$.

Proof. As $b_{X^{1-\ell}YX^\ell}(a_0, \dots, a_r) = b_Y(a_0 + \ell, a_1, \dots, a_r)$, the first assertion is an immediate consequence of Theorem 3.3.9. The second assertion is a consequence of the fact that if (G, V) is a PV of commutative parabolic type, the operators $X^{1-\ell}YX^\ell$ are (algebraically independent) generators of \mathcal{T}_0 (see Theorem 3.4.2 below). \square

3.4. The case of regular PV's of commutative parabolic type.

In the case where $(G, V) = (G, V^+)$ is a regular PV of commutative parabolic type, we obtain some specific results. We refer to section 2.4 for the notations and the structure of these spaces.

First of all remember that in this case the degree d_0 of Δ_0 is equal to $r + 1$ which is the rank of (G, V^+) as a MF space. Moreover the open G -orbit $\Omega^+ = \{x \in V^+ \mid \Delta_0(x) \neq 0\}$ is a symmetric space G/H where H is the isotropy subgroup of I^+ . Let \mathfrak{h} be the Lie algebra of H , and let \mathfrak{q} be the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to the Killing form of $\tilde{\mathfrak{g}}$. The space \mathfrak{q} can also be defined as the -1 eigenspace for the involution whose fixed point set is \mathfrak{h} . Let $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ be the set of strongly orthogonal roots occurring in the descent (see section 2.4) and let $\{H_{\alpha_0}, H_{\alpha_1}, \dots, H_{\alpha_r}\}$ be the corresponding set of co-roots. Define $\mathfrak{a} = \sum_{i=0}^r \mathbb{C}H_{\alpha_i}$. Then \mathfrak{a} is a maximal abelian subspace of \mathfrak{q} ([B-R], Proposition 5.4) and the dual space \mathfrak{a}^* can be identified with the space of restrictions of the fundamental characters $\lambda_0, \dots, \lambda_r$ to \mathfrak{a} . This is a consequence of [B-R], Lemme 2.5. Hence this definition of \mathfrak{a}^* is coherent with the direct definition ($\mathfrak{a}^* = \sum_{i=0}^r \mathbb{C}\lambda_i$) given in section 3.3. in the general case.

Lemma 3.4.1. *The root system $\Sigma(\tilde{\mathfrak{g}}, \mathfrak{a})$ is always of type C_{r+1} and the root system $\Sigma(\mathfrak{g}, \mathfrak{a}) = \Sigma$ is always of type A_r .*

Proof. Define

$$\tilde{E}_{ij}(k, \ell) = \{X \in \tilde{\mathfrak{g}} \mid [H_{\alpha_i}, X] = kX, [H_{\alpha_j}, X] = \ell X, [H_{\alpha_p}, X] = 0 \text{ if } p \neq i, j\}$$

and

$$\tilde{E}_i(k) = \{X \in \tilde{\mathfrak{g}} \mid [H_{\alpha_i}, X] = kX, [H_{\alpha_p}, X] = 0 \text{ if } p \neq i\}.$$

We know from Lemme 4.1. in [M-R-S] that:

$$V^+ = \bigoplus_{i < j} \tilde{E}_{ij}(1, 1) \oplus \bigoplus_{i=0}^r \tilde{E}_i(2)$$

and

$$\mathfrak{g} = \bigoplus_{i < j} \tilde{E}_{ij}(-1, 1) \oplus \bigoplus_{i=0}^r \tilde{E}_i(0) \oplus \bigoplus_{i < j} \tilde{E}_{ij}(1, -1)$$

Moreover one has $\bigoplus_{i=0}^r \tilde{E}_i(0) = \mathfrak{z}_{\tilde{\mathfrak{g}}}(\mathfrak{a})$. The preceding decompositions show that the spaces $\tilde{E}_{ij}(1, 1)$ and $\tilde{E}_i(2)$ are the root spaces of the pair $(\tilde{\mathfrak{g}}, \mathfrak{a})$. Let $\varepsilon_i = \frac{1}{2}\alpha_i$ be the dual basis of the basis H_{α_i} . Now it is clear that the positive roots of $\Sigma(\tilde{\mathfrak{g}}, \mathfrak{a})$ are the linear forms $\varepsilon_i + \varepsilon_j$ ($i < j$), $\varepsilon_i - \varepsilon_j$ ($i < j$) and $2\varepsilon_i$. This characterizes the root systems C_{r+1} and A_r . \square

Theorem 3.4.2.

Let (G, V^+) be a regular PV of commutative parabolic type.

- 1) For $\ell \in \mathbb{Z}$ set $D_\ell = X^{1-\ell} Y X^\ell$. Then D_0, D_1, \dots, D_r are algebraically independent generators of $\mathcal{T}_0 = D(V^+)^G$ (i.e. $\mathcal{T}_0 = \mathbb{C}[D_0, D_1, \dots, D_r]$).
- 2) We have $\mathcal{T} = D(\Omega^+)^{G'} = \mathbb{C}[X, X^{-1}, Y]$ where $\mathbb{C}[X, X^{-1}, Y]$ is the associative subalgebra of $D(\Omega^+)$ generated by X, X^{-1}, Y .
- 3) We have $\mathcal{T}_0[X, Y] = D(V)^{G'} = \mathbb{C}[X, Y, R_1, \dots, R_r]$ where the R_i 's are the Capelli operators which were introduced before Theorem 2.2.7, and where $\mathbb{C}[X, Y, R_1, \dots, R_r]$ is the associative subalgebra of $D(V^+)$ generated by X, Y, R_1, \dots, R_r .

Proof. 1) For $\lambda \in \mathfrak{t}^*$, we will denote by $\bar{\lambda}$ the restriction of λ to \mathfrak{a} . Through the "classical" Harish-Chandra isomorphism γ for symmetric spaces ([H-S], Part II, Theorem 4.3) the algebra \mathcal{T}_0 is isomorphic to $S(\mathfrak{a})^W = \mathbb{C}[\mathfrak{a}^*]^W$, where W is the Weyl group of the root system Σ of $(\mathfrak{g}, \mathfrak{a})$. From Lemma 3.4.1, this Weyl group is the symmetric group acting by permutations on the $\bar{\alpha}_i$'s. We will choose an order on Σ such that $\bar{R}^+ \subset \Sigma^+$. Define $\rho = \frac{1}{2} \sum_{\mu \in \Sigma^-} \mu$. It is well known that for $D \in \mathcal{T}_0$ and $\mu = \sum_{i=0}^r a_i \lambda_i \in \mathfrak{a}^*$, $\gamma(D)(\bar{\mu} - \rho)$ is equal to the eigenvalue of D acting on $\Delta_0^{a_0} \dots \Delta_r^{a_r}$. In other words we have:

$$\gamma(D)(\bar{\mu} - \rho) = b_D(\mu).$$

From [R-S-2], Lemme 3.9 p. 155 we know that

$$\rho = \frac{d}{4} \sum_{i < j} (\bar{\alpha}_i - \bar{\alpha}_j) = \frac{d}{4} \sum_{i=0}^r (r - 2i) \bar{\alpha}_i$$

and from [ibid.], Lemme 3.8 p. 155 we also have:

$$\bar{\mu} = a_0 \bar{\alpha}_0 + (a_0 + a_1) \bar{\alpha}_1 + \dots + (a_0 + \dots + a_r) \bar{\alpha}_r.^\dagger$$

Let us now make the following change of variables:

$$\mu_i = a_0 + \dots + a_i, \text{ for } i = 0, \dots, r.$$

As $b_{D_\ell}(\mu) = b_Y(\mu_0 + \ell, \dots, \mu_r + \ell) = c \prod_{i=0}^r (\mu_i + \ell + i \frac{d}{2})$ (see Example 3.1.2) we obtain

$$\begin{aligned} \gamma(D_\ell)(\bar{\mu}) &= b_{D_\ell}(\mu + \rho) = b_{D_\ell}(\sum_{i=0}^r \mu_i \bar{\alpha}_i + \frac{d}{4} \sum_{i=0}^r (r - 2i) \bar{\alpha}_i) \\ &= c \prod_{i=0}^r (\mu_i + \frac{d}{4} r + \ell). \end{aligned}$$

As expected the polynomials $\gamma(D_\ell)$ are symmetric (i.e. invariant under W). Moreover it is easy to prove that these polynomials, for $\ell = 0, \dots, r$, are algebraically independent generators of the algebra of symmetric polynomials. This proves 1).

2) As $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$ (see Proposition 3.1.7), and as, from 1), the elements of \mathcal{T}_0 are polynomials in X, X^{-1}, Y we obtain that $\mathcal{T} \subset \mathbb{C}[X, X^{-1}, Y]$. The inverse inclusion is obvious.

3) The inclusion $\mathbb{C}[X, Y, R_1, \dots, R_r] \subset D(V)^{G'} = \mathcal{T}_0[X, Y]$ is obvious. Conversely, from Theorem 2.2.7 we have $\mathcal{T}_0[X, Y] = \mathbb{C}[R_0, R_1, \dots, R_r][X, Y]$. As $R_0 = XY$ (see Remark 2.2.8), we have $\mathcal{T}_0[X, Y] \subset \mathbb{C}[X, Y, R_1, \dots, R_r]$.

[†]The change of sign with respect to Lemme 3.9 in [R-S-2] is due to the fact that we consider here characters of relative invariants instead of highest weights.

□

Remark 3.4.3. According to Terras [Ter](II, p.208), the operators D_ℓ were first considered by Selberg on positive definite symmetric matrices. They appear also in Maass ([M]), in the same context of positive definite symmetric matrices. In the context of symmetric cones, the analogue of assertion 1) of the preceding theorem can be found in [F-K](Corollary XIV.1.6).

Remark 3.4.4. Note first that in all cases $R_r = E$. In the special case where $G \simeq SO(k) \times \mathbb{C}^*$ and $V^+ \simeq \mathbb{C}^k$ (corresponding to the cases B_n and D_n^1 in Table 1), we have always $r = 1$, and assertion 3) of the preceding theorem yields

$$\mathbf{D}(\mathbb{C}^k)^{SO(k)} = \mathbb{C}[Q(x), Q(\partial), E]$$

where $Q(x) = X = \sum_{i=1}^k x_i^2$, $Q(\partial) = Y = \sum_{i=1}^k \frac{\partial^2}{\partial x_i^2}$.

This was proved by S. Rallis and G. Schiffmann ([Ra-S], Lemma 5.2. p. 112).

3.5. Ideals of $\mathcal{T} = \mathcal{T}_0[X, X^{-1}] = D(\Omega)^{G'}$ and $\mathcal{T}_0[X, Y] = D(V)^{G'}$.

Here (G, V) is again a MF space with a one dimensional quotient.

Let J be a left (resp. right) ideal of \mathcal{T} . Then J is said to be a graded left (resp. right) ideal if $J = \oplus_{i \in \mathbb{Z}} J_i$ where $J_i = J \cap \mathcal{T}_i$.

Theorem 3.5.1.

- 1) Let J be a graded left ideal of \mathcal{T} , then $J = \oplus_{i \in \mathbb{Z}} X^i J_0$. Conversely if J_0 is any ideal of the (commutative) algebra \mathcal{T}_0 , then $J = \oplus_{i \in \mathbb{Z}} X^i J_0$ is a graded left ideal of \mathcal{T} .
- 2) Let J be a graded right ideal of \mathcal{T} , then $J = \oplus_{i \in \mathbb{Z}} J_0 X^i$. Conversely if J_0 is any ideal of the (commutative) algebra \mathcal{T}_0 , then $J = \oplus_{i \in \mathbb{Z}} J_0 X^i$ is a graded right ideal of \mathcal{T} .
- 3) Let J be a two-sided ideal of \mathcal{T} . Then J is graded, J_0 is a τ -invariant ideal of \mathcal{T}_0 and $J = \oplus_{i \in \mathbb{Z}} X^i J_0 = \oplus_{i \in \mathbb{Z}} J_0 X^i$. Conversely if J_0 is a τ -invariant ideal of \mathcal{T}_0 , then $J = \oplus_{i \in \mathbb{Z}} X^i J_0 = \oplus_{i \in \mathbb{Z}} J_0 X^i$ is a two-sided ideal of \mathcal{T} .

Proof. Let $J = \oplus_{i \in \mathbb{Z}} J_i$ be a graded left ideal. Then J_0 is an ideal of \mathcal{T}_0 and $X^i J_0 \subset J \cap \mathcal{T}_i = J_i$ and conversely $X^{-i} J_i \subset J \cap \mathcal{T}_0 = J_0$. Therefore $J = \oplus_{i \in \mathbb{Z}} X^i J_0$.

If J_0 is an ideal of \mathcal{T}_0 , define $J = \oplus_{i \in \mathbb{Z}} X^i J_0$. Let $D_j \in \mathcal{T}_j$, then $D_j X^i J_0 = X^j X^i (X^{-i} X^{-j} D_j X^i) J_0$. As $(X^{-i} X^{-j} D_j X^i) \in \mathcal{T}_0$, we obtain that $D_j X^i J_0 \subset X^{i+j} J_0 \subset J$. Hence J is a graded left ideal.

The proof for graded right ideals is the same.

Let now J be a two-sided ideal. An element $D \in J$ can be written uniquely $D = \sum_{i=-\ell}^{\ell} D_i$, where $D_i \in \mathcal{T}_i$. As $E \in \mathcal{T}_0$, we have $[E, D] = ED - DE \in J$. Moreover $[E, D] = \sum_{i=-\ell}^{\ell} [E, D_i] = \sum_{i=-\ell}^{\ell} d_0 i D_i \in J$. By iterating the bracket with E we get:

$$k = 0, 1, \dots, 2\ell \quad (\text{ad } E)^k D = \sum_{i=-\ell}^{\ell} d_0^k i^k D_i \in J \quad (3-3-1)$$

The square matrix defined by the linear system $(3 - 3 - 1)$ is invertible because its determinant is a Van der Monde determinant. Therefore each operator D_i belongs to J . Hence J is graded.

As J is a two-sided ideal, we have $XJ_0X^{-1} \subset J_0$, hence J_0 is τ -invariant.

Applying part 1) and part 2) we see that $J = \oplus_{i \in \mathbb{Z}} X^i J_0 = \oplus_{i \in \mathbb{Z}} J_0 X^i$.

Conversely let J_0 be a τ -invariant ideal of \mathcal{T}_0 . Define $J = \oplus_{i \in \mathbb{Z}} X^i J_0$.

According to 1), J is a graded left ideal. But as J_0 is τ -stable one has $X^i J_0 X^{-i} = J_0$. Therefore $J = \oplus_{i \in \mathbb{Z}} (X^i J_0 X^{-i}) X^i = \oplus_{i \in \mathbb{Z}} J_0 X^i$. Then, according to 2), J is also a graded right ideal, hence two-sided. \square

Let us now give the description of the graded left or right ideals, and also the two-sided ideals of $\mathcal{T}_0[X, Y] = D(V)^{G'}$ which is a little bit more involved.

Theorem 3.5.2.

1) *The graded left ideals of $\mathcal{T}_0[X, Y] = D(V)^{G'}$ are exactly the subsets of the form*

$$J = (\oplus_{i > 0} Y^i U_{-i}) \oplus (\oplus_{i \geq 0} X^i V_i) \quad (3 - 3 - 2)$$

where the U_i 's and the V_i 's are ideals of \mathcal{T}_0 verifying the following conditions:

i) *There exists $n_0 \in \mathbb{N}$ such that*

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n_0} = V_{n_0+1} = V_{n_0+2} = \cdots \quad (3 - 3 - 3)$$

and such that

$$\tau^{-(i-1)}(YX)V_i \subseteq V_{i-1} \quad i = 1, \dots, n_0 \quad (3 - 3 - 4)$$

ii) *There exists $m_0 \in \mathbb{N}$ such that*

$$V_0 \subseteq U_{-1} \subseteq \cdots \subseteq U_{-m_0} = U_{-(m_0+1)} = U_{-(m_0+2)} = \cdots \quad (3 - 3 - 5)$$

and such that

$$\tau^{(i-1)}(XY)U_{-i} \subseteq U_{-(i-1)} \quad i = 1, \dots, m_0. \quad (3 - 3 - 6)$$

2) *The graded right ideals of $\mathcal{T}_0[X, Y] = D(V)^{G'}$ are exactly the subsets of the form*

$$J = (\oplus_{i > 0} U_{-i} Y^i) \oplus (\oplus_{i \geq 0} V_i X^i)$$

where the U_i 's and the V_i 's are ideals of \mathcal{T}_0 verifying the following conditions:

i) *There exists $n_0 \in \mathbb{N}$ such that*

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n_0} = V_{n_0+1} = V_{n_0+2} = \cdots$$

and such that

$$V_i \tau^{(i-1)}(XY) \subseteq V_{i-1} \quad i = 1, \dots, n_0$$

ii) *There exists $m_0 \in \mathbb{N}$ such that*

$$V_0 \subseteq U_{-1} \subseteq \cdots \subseteq U_{-m_0} = U_{-(m_0+1)} = \cdots$$

and such that

$$U_{-i} \tau^{-(i-1)}(YX) \subseteq U_{-(i-1)} \quad i = 1, \dots, m_0.$$

Proof. Let J be a graded left ideal of $\mathcal{T}_0[X, Y]$. Write $J = \oplus J_i$, with $J_i = J \cap \mathcal{T}_i$ (see (3-1-2)). It is easy to see that J_0 is an ideal of \mathcal{T}_0 . More generally let us set, for $i \geq 0$, $V_i = \{v \in \mathcal{T}_0, X^i v \in J_i\}$ and $U_{-i} = \{u \in \mathcal{T}_0, Y^i u \in J_{-i}\}$, (hence $V_0 = U_0 = J_0$). Again it is easy to see that the V_i 's and the U_{-i} 's are ideals in \mathcal{T}_0 . From Proposition 3.1.7 we know that $J_i = X^i V_i$ if $i \geq 0$, and $J_i = Y^i V_i$ if $i < 0$, hence $J = (\oplus_{i>0} Y^i U_{-i}) \oplus (\oplus_{i \geq 0} X^i V_i)$. If $v \in V_i$, then $X^i v \in J_i$, and therefore $X(X^i v) = X^{i+1} v \in J_{i+1}$. Hence $v \in V_{i+1}$, and $V_i \subseteq V_{i+1}$. Similarly one proves that $U_{-i} \subseteq U_{-(i+1)}$.

We have obtained two increasing sequences of ideals of \mathcal{T}_0 . As \mathcal{T}_0 is polynomial algebra from Theorem 2.2.7, it is noetherian and hence there exist n_0 and m_0 such that

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n_0} = V_{n_0+1} = V_{n_0+2} = \cdots$$

$$U_0 \subseteq U_{-1} \subseteq \cdots \subseteq U_{-m_0} = U_{-(m_0+1)} = U_{-(m_0+2)} = \cdots$$

and hence relations (3-3-3) and (3-3-5) are proved.

If we take such two sequences of ideals of \mathcal{T}_0 (satisfying the relations (3-3-3) and (3-3-5)) and if $D \in \mathcal{T}_0$, we have $DY^i U_{-i} = Y^i \tau^i(D) U_{-i} \subseteq Y^i U_{-i}$. Similarly $DX^i V_i = X^i \tau^{-i}(D) V_i \subseteq X^i V_i$. Therefore if we set

$$J = (\oplus_{i>0} Y^i U_{-i}) \oplus (\oplus_{i \geq 0} X^i V_i),$$

then J is invariant under left multiplications by elements of \mathcal{T}_0 .

In order, for such a J to be a graded left ideal, it is now sufficient to be invariant under left multiplication by X and Y . As the sequences U_{-i} and V_i are increasing we have $Y(Y^i U_{-i}) = Y^{i+1} U_{-i} \subseteq Y^{i+1} U_{-(i+1)}$ and $X(X^i V_i) = X^{i+1} V_i \subseteq X^{i+1} V_{i+1}$. But we need also to have $X(Y^i U_{-i}) \subseteq Y^{i-1} U_{-(i-1)}$. This condition can be written $X(Y^i U_{-i}) = (XY)Y^{i-1} U_{-i} = Y^{i-1} \tau^{i-1}(XY) U_{-i} \subseteq Y^{i-1} U_{-(i-1)}$. As $\mathcal{T}_0[X, Y]$ has no zero divisors this implies that $\tau^{i-1}(XY) U_{-i} \subseteq U_{-(i-1)}$, which is condition (3-3-6). And we need also to have $YX^i V_i \subseteq X^{i-1} V_{i-1}$. This condition can be written $YX^i V_i = (YX)X^{i-1} V_i = X^{i-1} \tau^{-(i-1)}(YX) V_i \subseteq X^{i-1} V_{i-1}$. This implies that $\tau^{-(i-1)}(YX) V_i \subseteq V_{i-1}$, which is condition (3-3-4).

Conversely if conditions (3-3-4) and (3-3-6) are satisfied, it is easy to see that the set

$$J = (\oplus_{i>0} Y^i U_{-i}) \oplus (\oplus_{i \geq 0} X^i V_i)$$

is stable under left multiplications by X and Y . Hence J is a graded left ideal. This proves the first part of the theorem.

The proof for the graded right ideals is similar. □

Let us now describe the ideals (i.e. two sided ideals) of $\mathcal{T}_0[X, Y] = D(V)^{G'}$.

Theorem 3.5.3.

- 1) Let J be a two-sided ideal of $\mathcal{T}_0[X, Y]$. Then J is graded.
- 2) The two-sided ideals of $\mathcal{T}_0[X, Y]$ are exactly the subsets of the form

$$J = (\oplus_{i>0} Y^i U_{-i}) \oplus U_0 \oplus (\oplus_{i>0} X^i V_i)$$

where the U_i 's and the V_i 's are ideals of \mathcal{T}_0 verifying the following conditions:

- i) There exists $n_0 \in \mathbb{N}$ such that

$$V_0 = U_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n_0} = V_{n_0+1} = V_{n_0+2} = \cdots$$

and there exists $m_0 \in \mathbb{N}$ such that

$$V_0 = U_0 \subseteq U_{-1} \subseteq \cdots \subseteq U_{-m_0} = U_{-(m_0+1)} = U_{-(m_0+2)} = \cdots$$

Moreover U_{-m_0} and V_{n_0} are τ -stable:

$$U_{-m_0} = \tau(U_{-m_0}), \quad V_{n_0} = \tau(V_{n_0})$$

ii) If $U'_{-i} = \tau^{-i}(U_{-i})$ and $V'_i = \tau^i(V_i)$, then

$$V'_0 \subseteq V'_1 \subseteq \cdots \subseteq V'_{n_0} = V_{n_0} = V'_{n_0+1} = V'_{n_0+2} = \cdots$$

$$U'_0 = V'_0 \subseteq U'_{-1} \subseteq \cdots \subseteq U'_{-m_0} = U_{m_0} = U'_{-(m_0+1)} = U'_{-(m_0+2)} = \cdots$$

iii)

$$\begin{aligned} \tau^{i-1}(XY)U_{-i} &\subseteq U_{-(i-1)} & i = 1, \dots, m_0 \\ \tau^{-(i-1)}(YX)V_i &\subseteq V_{i-1} & i = 1, \dots, n_0 \end{aligned}$$

iv)

$$\begin{aligned} \tau^{-1}(U_{-i})YX &\subseteq U_{-(i-1)} & i = 1, \dots, m_0 \\ \tau(V_i)XY &\subseteq V_{i-1} & i = 1, \dots, n_0 \end{aligned}$$

Proof. 1) The proof that a two-sided ideal of $\mathcal{T}_0[X, Y]$ is graded is the same as for assertion 3) of Theorem 3.5.1.

2) Let J be a two-sided ideal of $\mathcal{T}_0[X, Y]$. Let us first consider it as a left graded ideal. Then from Theorem 3.5.2 we can write:

$$J = (\oplus_{i>0} Y^i U_{-i}) \oplus U_0 \oplus (\oplus_{i>0} X^i V_i)$$

where the ideals U_{-i} and V_i verify conditions (3 – 3 – 3) to (3 – 3 – 6). In particular they are increasing, and stationary from the index m_0 and n_0 respectively.

We set now for all $i \in \mathbb{N}$:

$$U'_{-i} = \tau^{-i}(U_{-i}), \quad V'_i = \tau^i(V_i).$$

The U'_{-i} 's and the V'_i 's are again ideals of \mathcal{T}_0 . Moreover as $Y^i U_{-i} = U'_{-i} Y^i$ and $X^i V_i = V'_i X^i$ we obtain that

$$J = (\oplus_{i>0} U'_{-i} Y^i) \oplus U_0 \oplus (\oplus_{i>0} V'_i X^i).$$

As J is also a graded right ideal, we know from the second part of Theorem 3.5.2, that the sequences of ideals U'_{-i} and V'_i must be increasing. Let us first consider the V'_i 's. We have:

$$\begin{aligned} V'_0 \subseteq V'_1 = \tau(V_1) \subseteq V'_2 = \tau^2(V_2) \subseteq \cdots \subseteq V'_{n_0} = \tau^{n_0}(V_{n_0}) \\ \subseteq V'_{n_0+1} = \tau^{n_0+1}(V_{n_0+1}) \subseteq \cdots \end{aligned}$$

But as the sequence V_{n_0} is stationnary from the index n_0 , we have

$$\tau^{n_0}(V_{n_0}) \subseteq \tau^{n_0+1}(V_{n_0+1}) = \tau^{n_0+1}(V_{n_0}) \subseteq \cdots \subseteq \tau^{n_0+p}(V_{n_0}) \subseteq \cdots$$

and as \mathcal{T}_0 is noetherian, there exists $p \geq n_0$ such that $\tau^p(V_{n_0}) = \tau^{p+1}(V_{n_0})$. Hence $V_{n_0} = \tau(V_{n_0})$, that is V_{n_0} is τ -stable. This implies that $V'_{n_0} = V_{n_0}$ and that the sequence V'_i is stationnary from the same index n_0 . A similar proof shows that the sequence U'_{-i} is stationary from the index m_0 , that $U'_{-m_0} = U_{m_0}$ and that U_{m_0} is τ -stable.

At this point we have proved that if

$$J = (\oplus_{i>0} Y^i U_{-i}) \oplus U_0 \oplus (\oplus_{i>0} X^i V_i),$$

then assertions 2) *i*) and 2) *ii*) are true.

But they are still other conditions which are needed for J to be a two-sided ideal, namely

$$\begin{aligned} \tau^{-(i-1)}(YX)V_i &\subseteq V_{i-1} && \text{for } i = 1, \dots, n_0 \\ \tau^{i-1}(XY)U_{-i} &\subseteq U_{-(i-1)} && \text{for } i = 1, \dots, m_0 \\ V'_i \tau^{i-1}(XY) &\subseteq V'_{i-1} && \text{for } i = 1, \dots, n_0 \\ U'_{-i} \tau^{-(i-1)}(YX) &\subseteq U'_{-(i-1)} && \text{for } i = 1, \dots, m_0 \end{aligned}$$

(see Theorem 3.5.2).

But as $U'_{-i} = \tau^{-i}(U_{-i})$ and $V'_i = \tau^i(V_i)$, the two last inclusions can be written

$$\tau^i(V_i) \tau^{i-1}(XY) \subseteq \tau^{i-1}(V_{i-1}).$$

$$\tau^{-i}(U_{-i}) \tau^{-(i-1)}(YX) \subseteq \tau^{-(i-1)}(U_{-(i-1)})$$

Applying respectively $\tau^{-(i-1)}$ and τ^{i-1} to these inclusions we obtain

$$\tau(V_i)XY \subseteq V_{i-1}, \text{ for } i = 1, \dots, n_0$$

$$\tau^{-1}(U_{-i})YX \subseteq U_{-(i-1)}, \text{ for } i = 1, \dots, m_0.$$

Hence conditions 2) *iii*) and 2) *iv*) are proved.

Conversely we have to prove that if

$$J = (\oplus_{i>0} Y^i U_{-i}) \oplus U_0 \oplus (\oplus_{i \geq 0} X^i V_i),$$

where the ideals U_{-i} and V_i of \mathcal{T}_0 verify the conditions 2) *i*), 2) *ii*), 2) *iii*) and 2) *iv*) above, then J is a two-sided ideal of $\mathcal{T}_0[X, Y]$. This is just a straightforward verification. \square

Example 3.5.4. Let V a τ -stable ideal of \mathcal{T}_0 . Then

$$J(V) = (\oplus_{i>0} Y^i V) \oplus V \oplus (\oplus_{i \geq 0} X^i V)$$

is a two-sided ideal of $\mathcal{T}_0[X, Y]$.

Theorem 3.5.1 and Theorem 3.5.3 show the importance of τ -invariant ideals of \mathcal{T}_0 in the description of the two-sided ideals of $\mathcal{T}_0[X, X^{-1}] = D(\Omega)^{G'}$ and $\mathcal{T}_0[X, Y] = D(V)^{G'}$. For completeness we indicate how such ideals are obtained.

Proposition 3.5.5. *Any τ -invariant ideal of \mathcal{T}_0 ($= D(V)^G = D(\Omega)^G$) is generated by a finite number of τ -invariant polynomials.*

Proof. Remember from Theorem 3.3.1 that $\mathcal{T}_0 = D(V)^G$ is isomorphic through the Harish-Chandra isomorphism to $\mathbb{C}[A]^{W_0}$ where W_0 is a finite reflection subgroup of W . Therefore we have to prove that any τ -invariant ideal of $\mathbb{C}[A]^{W_0}$ is generated by a finite number of τ -invariant polynomials, where the automorphism τ of $\mathbb{C}[A]^{W_0}$ is defined for $P \in \mathbb{C}[A]^{W_0}$ by $\tau(P)(\lambda + \rho) = P((\lambda - \lambda_0) + \rho)$ (see Definition 3.3.2). Let $\lambda = a_0 \lambda_0 + \dots + a_r \lambda_r \in \mathfrak{a}^*$, then any $P \in \mathbb{C}[A]$ can be viewed as a polynomial in a_0, a_1, \dots, a_r , and then $\tau(P)(a_0, a_1, \dots, a_r) = P(a_0 - 1, a_1, \dots, a_r)$. Let $P \in \mathbb{C}[A]$ and let d be the

degree of P in a_0 . In the ring $\mathbb{C}[A] = \mathbb{C}[a_0, a_1, \dots, a_r] = \mathbb{C}[a_1, \dots, a_r][a_0]$ we can write:

$$P(a_0, \dots, a_r) = P_d(a_1, \dots, a_r)a_0^d + P_{d-1}(a_1, \dots, a_r)a_0^{d-1} + \dots \quad (*)$$

Let us show by induction that P_d is a linear combination of $P, \tau(P), \tau^2(P), \dots$. It is certainly true if $d = 0$, because in that case $P_0 = P$. Suppose that it is true for all degrees $\leq d-1$. Then

$$P - \tau(P) = P_d(a_1, \dots, a_r)(a_0^d - (a_0 - 1)^d) + \dots$$

Hence the degree in a_0 of $P - \tau(P)$ is $d-1$, and therefore by the induction hypothesis we obtain that P_d is a linear combination of $P - \tau(P), \tau(P - \tau(P)), \tau^2(P - \tau(P)), \dots$, hence a linear combination of $P, \tau(P), \tau^2(P), \dots$. Remark also that the polynomials P_0, P_1, \dots, P_d are τ -invariant because they do not depend on a_0 .

It follows that if I is τ -invariant ideal of $\mathbb{C}[A]^{W_0}$ and if $P \in I$, then all the coefficients $P_i(a_1, \dots, a_r)$ in the expansion $(*)$ above belong to I . As $h(E)(\lambda + \rho) = a_0 d_0 + a_1 d_1 + \dots + a_r d_r$, the polynomial $\sigma_1(a_0, a_1, \dots, a_r) = a_0 d_0 + a_1 d_1 + \dots + a_r d_r$ belongs to $\mathbb{C}[A]^{W_0}$. Then $P - P_d(\frac{\sigma_1}{d_0})^d$ belongs still to I , and its degree in a_0 is $< d$. An easy induction on d shows that any $P \in I$ can be written $P = \sum P_j \sigma_j$, where P_j belongs to I and is τ -invariant, and where $\sigma_j \in \mathbb{C}[A]^{W_0}$. This proves that the ideal $I \subset \mathbb{C}[A]^{W_0}$ is generated by a finite number of τ -invariant polynomials. \square

4. EMBEDDING OF \mathcal{T} INTO $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_r]$.

In this section again (G, V) is a multiplicity free space with a one dimensional quotient. Remember that we have defined $\Omega = \{x \in V \mid \Delta_0(x) \neq 0\}$.

4.1. Vector space embedding of $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ into $End(\mathbb{C}[\Omega])$.

Recall from Lemma 2.2.3 the multiplicity free decomposition of $\mathbb{C}[\Omega]$ into irreducible representations of G :

$$\mathbb{C}[\Omega] = \bigoplus_{\mathbf{a}=(a_0, a_1, \dots, a_r) \in \mathbb{Z} \times \mathbb{N}^r} V_{\mathbf{a}}.$$

For $\mathbf{a} = (a_0, a_1, \dots, a_r)$ and $m \in \mathbb{Z}$, we set $\mathbf{a} + m = (a_0 + m, a_1, \dots, a_r)$. Then for $P \in V_{\mathbf{a}}$ we have

$$XP = \Delta_0 P \in V_{\mathbf{a}+1} \quad (4-1-1)$$

$$YP = b_Y(\mathbf{a}) \frac{P}{\Delta_0} = b_Y(\mathbf{a}) X^{-1} P \in V_{\mathbf{a}-1} \quad (4-1-2)$$

$$EP = b_E(\mathbf{a}) P \in V_{\mathbf{a}} \quad (4-1-3)$$

(Remember that $b_E(\mathbf{a}) = d_0 a_0 + d_1 a_1 + \dots + d_r a_r$ is the degree of the polynomials in $V_{\mathbf{a}}$).

Therefore the operators E, X, Y, X^{-1} are very well understood as elements of $End(\mathbb{C}[\Omega])$ (Here $End(\mathbb{C}[\Omega])$ denotes the set of linear mappings from $\mathbb{C}[\Omega]$

into itself). More precisely the operator E is diagonalisable on $\mathbb{C}[\Omega]$. The action of Y "goes down" from $V_{\mathbf{a}}$ to $V_{\mathbf{a}-1}$. More precisely the action of Y on $V_{\mathbf{a}}$ is the action of X^{-1} (division by Δ_0) multiplied by the value of the polynomial b_Y at \mathbf{a} .

The preceding remarks suggest an embedding of the vector space $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ into $\text{End}(\mathbb{C}[\Omega])$. Let us make this more precise. If U and V are vector spaces over \mathbb{C} we will denote by $\mathcal{L}(U, V)$ the vector space of linear maps from U to V .

Let us also remark that from the preceeding decomposition of $\mathbb{C}[\Omega]$ we get:

$$\text{End}(\mathbb{C}[\Omega]) = \bigoplus_{\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^n} \mathcal{L}(V_{\mathbf{a}}, \mathbb{C}[\Omega]) \quad (4-1-4)$$

Definition 4.1.1. For $\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^r$, $P \in \mathbb{C}[X_0, X_1, \dots, X_r]$, $q \in \mathbb{C}[t, t^{-1}]$ we will denote by $\varphi_{\mathbf{a}}(q \otimes P)$ the element of $\mathcal{L}(V_{\mathbf{a}}, \mathbb{C}[\Omega])$ defined by:

$$Q_{\mathbf{a}} \in V_{\mathbf{a}}, \quad \varphi_{\mathbf{a}}(q \otimes P)Q_{\mathbf{a}} = P(\mathbf{a})q(\Delta_0)Q_{\mathbf{a}}.$$

($\varphi_{\mathbf{a}}$ defines a linear map from $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ into $\mathcal{L}(V_{\mathbf{a}}, \text{End}(\mathbb{C}[\Omega]))$)
From (4-1-4) we define then an element

$$\varphi \in \mathcal{L}(\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r], \text{End}(\mathbb{C}[\Omega])) \text{ by}$$

$$\varphi = \bigoplus_{\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^r} \varphi_{\mathbf{a}}.$$

Proposition 4.1.2.

The linear map :

$$\varphi : \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n] \longrightarrow \text{End}(\mathbb{C}[\Omega])$$

is injective and its image is stable under the multiplication (composition of mappings) in $\text{End}(\mathbb{C}[\Omega])$.

Proof. Any element u in $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ can be written as a finite sum

$$u = \sum t^i \otimes P_i \quad i \in \mathbb{Z}, P_i \in \mathbb{C}[X_0, \dots, X_r].$$

Suppose that $u \in \ker \varphi$. Then for any $Q_{\mathbf{a}} \in V_{\mathbf{a}}$ we have:

$$0 = \varphi(u)Q_{\mathbf{a}} = \sum \varphi(t^i \otimes P_i)Q_{\mathbf{a}} = \sum P_i(\mathbf{a})\Delta_0^i Q_{\mathbf{a}}.$$

This implies that $P_i(\mathbf{a}) = 0$ for all i and all \mathbf{a} , and therefore φ is injective. To prove the stability under multiplication it is enough to prove that for any $R, S \in \mathbb{C}[X_0, \dots, X_r]$ and $\ell, m \in \mathbb{Z}$ the endomorphism $\varphi(t^m \otimes R)\varphi(t^\ell \otimes S)$ belongs to the image of φ . If $Q_{\mathbf{a}} \in V_{\mathbf{a}}$, we have $\varphi(t^m \otimes R)\varphi(t^\ell \otimes S)Q_{\mathbf{a}} = \varphi(t^m \otimes R)S(\mathbf{a})\Delta_0^\ell Q_{\mathbf{a}} = R(\mathbf{a} + \ell)S(\mathbf{a})\Delta_0^{m+\ell} Q_{\mathbf{a}} = \varphi(t^{m+\ell} \otimes (\tau_\ell R)S)Q_{\mathbf{a}}$ where $\tau_\ell R(X) = R(X + \ell)$ and where $X + \ell = (X_0 + \ell, X_1, \dots, X_r)$. □

4.2. Algebra embedding of $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_r]$ into $End(\mathbb{C}[\Omega])$.

Recall that if \mathbf{A} and \mathbf{B} are two associative algebras the tensor product $\mathbf{A} \otimes \mathbf{B}$ is again an algebra, called the tensor product algebra, with the multiplication defined by

$$a_1, a_2 \in \mathbf{A}, b_1, b_2 \in \mathbf{B} \quad (a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2 \quad (4-2-1)$$

The tensor product algebra $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ is commutative. On the other hand the algebra $End(\mathbb{C}[\Omega^+])$ is of course non commutative. Let us use the injectivity of φ and the stability of its image under multiplication to define a new multiplication in $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ by setting for $q, r \in \mathbb{C}[t, t^{-1}]$ and for $P, Q \in \mathbb{C}[X_0, \dots, X_r]$:

$$(q \otimes P)(r \otimes Q) = \varphi^{-1}(\varphi(q \otimes P)\varphi(r \otimes Q)) \quad (4-2-2)$$

More explicitly it is easy to see that for $m, l \in \mathbb{Z}$ and $P, Q \in \mathbb{C}[X_0, \dots, X_r]$ we have:

$$(t^m \otimes P)(t^l \otimes Q) = t^{m+l} \otimes (\tau_l P)Q \quad (4-2-3)$$

(where $\tau_l P(X_0, X_1, \dots, X_r) = P(X_0 + l, X_1, \dots, X_r)$).

With this multiplication $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ becomes a non commutative associative algebra.

We will denote by $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] = \mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ the algebra of differential operators on \mathbb{C}^* whose coefficients are Laurent polynomials. In other words we have $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] = D(\mathbb{C}^*)$, the Weyl algebra of the torus.

Proposition 4.2.1.

The algebra $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ whose multiplication is defined by (4-2-3) is isomorphic to the tensor product algebra $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_r]$ (extended Weyl algebra of the torus).

Proof. We define first an isomorphism

$$\nu : \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r] \longrightarrow \mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_r]$$

between the underlying vector spaces. After that we will prove that ν is in fact an algebra isomorphism.

A vector basis of $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ is given by the elements $t^m \otimes X_0^{\alpha_0} X_1^{\alpha_1} \dots X_r^{\alpha_r}$ ($m \in \mathbb{Z}, \alpha_i \in \mathbb{N}$). The isomorphism ν is defined by

$$\nu(t^m \otimes X_0^{\alpha_0} X_1^{\alpha_1} \dots X_r^{\alpha_r}) = t^m (t \frac{d}{dt})^{\alpha_0} \otimes X_1^{\alpha_1} \dots X_r^{\alpha_r} \quad (4-2-4)$$

(As the elements $t^m (t \frac{d}{dt})^{\alpha_0} \otimes X_1^{\alpha_1} \dots X_r^{\alpha_r}$ define a vector basis of $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_r]$ the linear map ν is effectively a vector space isomorphism).

If $\mathbf{X} = (X_0, \dots, X_r)$, we set $\tilde{\mathbf{X}} = (X_1, \dots, X_r)$. In order to prove that ν is an isomorphism of algebras, it is enough to prove that for any $m, l \in \mathbb{Z}, i, j \in \mathbb{N}$ and for any $A, B \in \mathbb{C}[X_1, \dots, X_r]$,

$$\nu[(t^m \otimes X_0^i A(\tilde{\mathbf{X}}))(t^l \otimes X_0^j B(\tilde{\mathbf{X}}))] = \nu[t^m \otimes X_0^i A(\tilde{\mathbf{X}})] \nu[t^l \otimes X_0^j B(\tilde{\mathbf{X}})] \quad (4-2-5)$$

where the product on the left side is defined by (4-2-3) in $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_r]$ and the second one is the usual product in the tensor algebra $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_r]$.

We will need the following lemma whose easy proof by induction is left to the reader.

Lemma 4.2.2. *For any $i, j \in \mathbb{N}$, $\ell \in \mathbb{Z}$:*

$$(t \frac{d}{dt})^i t^\ell (t \frac{d}{dt})^j = \sum_{p=0}^i \binom{i}{p} \ell^{i-p} t^\ell (t \frac{d}{dt})^{p+j}.$$

Consider the left hand side of (4 – 2 – 5):

$$\begin{aligned} & \nu[(t^m \otimes X_0^i A(\tilde{\mathbf{X}}))(t^\ell \otimes X_0^j B(\tilde{\mathbf{X}}))] = \nu[t^{m+\ell} \otimes (X_0 + \ell)^i X_0^j A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}})] \\ &= \sum_{p=0}^i \nu[t^{m+\ell} \otimes \binom{i}{p} \ell^{i-p} X_0^{p+j} A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}})] \\ &= \sum_{p=0}^i t^{m+\ell} \binom{i}{p} \ell^{i-p} (t \frac{d}{dt})^{p+j} \otimes A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}}). \end{aligned}$$

On the right hand side of (4 – 2 – 5) we have:

$$\nu[t^m \otimes X_0^i A(\tilde{\mathbf{X}})] \nu[t^\ell \otimes X_0^j B(\tilde{\mathbf{X}})] = t^m (t \frac{d}{dt})^i t^\ell (t \frac{d}{dt})^j \otimes A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}}) \text{ and then using Lemma 4.2.2 we obtain the equality.}$$

□

Let us summarize the preceding results:

Theorem 4.2.3.

1) *The linear map*

$$\Psi : \mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_r] \longrightarrow \text{End}(\mathbb{C}[\Omega])$$

defined for $Q_{\mathbf{a}} \in V_{\mathbf{a}}$ by

$$\Psi(t^m (t \frac{d}{dt})^{\alpha_0} \otimes P(X_1, \dots, X_r)) Q_{\mathbf{a}} = a_0^{\alpha_0} P(a_1, \dots, a_r) \Delta_0^m Q_{\mathbf{a}}$$

is an injective homomorphism of associative algebras.

2) *The algebra $\mathcal{T} = \mathcal{T}_0[X, X^{-1}] = D(\Omega)^{G'}$ is in the image of Ψ . More precisely:*

a) *If $D \in \mathcal{T}_p$, then $D = \Psi(t^p b_D(t \frac{t}{dt}, X_1, \dots, X_r))$.*

b) *In particular*

$$E = \Psi(d_0(t \frac{d}{dt}) + d_1 X_1 + \dots + d_r X_r), \quad X = \Psi(t), \quad X^{-1} = \Psi(t^{-1}),$$

$$Y = \Psi(t^{-1} b_Y(t \frac{t}{dt}, X_1, \dots, X_r)).$$

5. THE STRUCTURE OF $D(V)^{G'}$

5.1. Smith algebras over rings.

Definition 5.1.1. *Let \mathbf{A} be a commutative associative algebra over \mathbb{C} , with unit element 1 and without zero divisors. Let $f, u \in \mathbf{A}[t]$ be two polynomials in one variable with coefficients in \mathbf{A} . Let $n \in \mathbb{N}^*$.*

- 1) The Smith algebra $S(\mathbf{A}, f, n)$ is the associative algebra over \mathbf{A} with generators (x, y, e) subject to the relations $[e, x] = nx$, $[e, y] = -ny$, $[y, x] = f(e)$.
 2) The algebra $U(\mathbf{A}, u, n)$ is the associative algebra over \mathbf{A} with generators $(\tilde{x}, \tilde{y}, \tilde{e})$ subject to the relations $[\tilde{e}, \tilde{x}] = n\tilde{x}$, $[\tilde{e}, \tilde{y}] = -n\tilde{y}$, $\tilde{x}\tilde{y} = u(\tilde{e})$, $\tilde{y}\tilde{x} = u(\tilde{e} + n)$.

Remark 5.1.2. 1) The algebras $S(\mathbb{C}, f, n)$ were introduced and intensively studied by S. P. Smith ([Sm]) who called them "algebras similar to $\mathcal{U}(\mathfrak{sl}_2)$ ", where $\mathcal{U}(\mathfrak{sl}_2)$ is the enveloping algebra of \mathfrak{sl}_2 . In fact they share many interesting properties with $\mathcal{U}(\mathfrak{sl}_2)$, in particular they have a very rich representation theory.

2) In fact we will see below that the algebra $U(\mathbf{A}, f, n)$ is a quotient of $S(\mathbf{A}, f, n)$.

3) One can prove, as in [Sm], that if the degree of f is one and $n \neq 0$, and if the leading coefficient is invertible in \mathbf{A} , then $S(\mathbf{A}, f, n)$ is isomorphic to the enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\mathbf{A}))$.

For a ring S , an automorphism $\sigma \in \text{Aut}(S)$ and a σ -derivation δ of S , the skew polynomial algebra $S[t, \sigma, \delta]$ has been defined in section 3.2.

Proposition 5.1.3.

Let \mathfrak{b} the 2-dimensional Lie algebra over \mathbf{A} , with basis $\{\varepsilon, \alpha\}$ and relation $[\varepsilon, \alpha] = n\alpha$. Let $\mathcal{U}(\mathfrak{b})$ be the enveloping algebra of \mathfrak{b} . Define an automorphism σ of $\mathcal{U}(\mathfrak{b})$ by $\sigma(\alpha) = \alpha$ and $\sigma(\varepsilon) = \varepsilon - n$ and define also a σ -derivation δ of $\mathcal{U}(\mathfrak{b})$ by $\delta(\alpha) = f(\varepsilon)$ and $\delta(\varepsilon) = 0$. Then $S(\mathbf{A}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$.

Proof. The proof is almost the same as the one given by S. P. Smith ([Sm], Prop. 1.2.). It suffices to remark that the algebra $\mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$ is an algebra over \mathbf{A} with generators ε, α, t subject to the relations

$$\begin{aligned} [\varepsilon, \alpha] &= n\alpha, \\ \alpha t &= t\sigma(\alpha) + \delta(\alpha) \text{ which is equivalent to } \alpha t = t\alpha + f(\varepsilon), \\ \varepsilon t &= t\sigma(\varepsilon) + \delta(\varepsilon) \text{ which is equivalent to } \varepsilon t = t(\varepsilon - n) = t\varepsilon - nt. \end{aligned}$$

Then the isomorphism $S(\mathbf{A}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$ is given by $e \mapsto \varepsilon$, $x \mapsto \alpha$ and $y \mapsto t$. □

Corollary 5.1.4.

$S(\mathbf{A}, f, n)$ is a noetherian domain with \mathbf{A} -basis $\{y^i x^j e^k, i, j, k \in \mathbb{N}\}$ (or any similar family of ordered monomials obtained by permutation of the elements (y, x, e)).

Proof. (compare with [Sm], proof of corollary 1.3 p. 288). We know from [MC-R] Th.1.2.9, that as $\mathcal{U}(\mathfrak{b})$ is a noetherian domain, so is $S(\mathbf{A}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$. Since

$$\begin{aligned} \mathcal{U}(\mathfrak{b})[t, \sigma, \delta] &= \mathcal{U}(\mathfrak{b}) \oplus \mathcal{U}(\mathfrak{b})t \oplus \mathcal{U}(\mathfrak{b})t^2 \oplus \mathcal{U}(\mathfrak{b})t^3 \oplus \cdots \oplus \mathcal{U}(\mathfrak{b})t^\ell \oplus \cdots \\ &= \mathcal{U}(\mathfrak{b}) \oplus t\mathcal{U}(\mathfrak{b}) \oplus t^2\mathcal{U}(\mathfrak{b}) \oplus t^3\mathcal{U}(\mathfrak{b}) \oplus \cdots \oplus t^\ell\mathcal{U}(\mathfrak{b}) \oplus \cdots \end{aligned}$$

(direct sums of \mathbf{A} -modules) and since the Poincaré-Birkhoff-Witt Theorem is still true for enveloping algebras of Lie algebras which are free over rings (see [Bou-1]), the ordered monomials in (y, x, e) beginning or ending with

y form a basis of the algebra $S(\mathbf{A}, f, n)$. To obtain the basis $\{e^i y^j x^k\}$ or $\{x^k y^j e^i\}$ it suffices to replace the algebra \mathfrak{b} by the algebra \mathfrak{b}_- which is generated by e and y . \square

Remark 5.1.5. The adjoint action of e ($u \mapsto [e, u]$) on $S(\mathbf{A}, f, n)$ is semi-simple and gives a decomposition of $S(\mathbf{A}, f, n)$ into weight spaces:

$$S(\mathbf{A}, f, n) = \bigoplus_{\nu \in \mathbb{Z}} S(\mathbf{A}, f, n)^\nu$$

where $S(\mathbf{A}, f, n)^\nu = \{u \in S(\mathbf{A}, f, n), [e, u] = \nu nu\}$. As $[e, x^j y^i e^k] = n(j - i)y^i x^j e^k$, we obtain, using Corollary 5.1.4, that the ordered monomials of the form $x^i y^j e^k$ form an \mathbf{A} -basis for $S(\mathbf{A}, f, n)^0$. Moreover as $yx = xy + f(e)$, it is easy to see that $S(\mathbf{A}, f, n)^0 = \mathbf{A}[xy, e] = \mathbf{A}[yx, e]$, where $\mathbf{A}[xy, e]$ (resp. $\mathbf{A}[yx, e]$) denotes the \mathbf{A} -subalgebra generated by xy (resp. yx) and e .

The proof of the following Lemma is straightforward.

Lemma 5.1.6.

Let $n \in \mathbb{N}^*$ and let $f \in \mathbf{A}[t]$. There exists an element $u \in \mathbf{A}[t]$, which is unique up to addition of an element of \mathbf{A} , such that

$$f(t) = u(t + n) - u(t) \quad (5-1-1)$$

Proposition 5.1.7. (compare with [Sm], Prop. 1.5)

Let u be as in the preceding Lemma. Define

$$\Omega_1 = xy - u(e) \text{ and } \Omega_2 = xy + yx - u(e + n) - u(e).$$

Then $\Omega_2 = 2\Omega_1$ and the center of $S(\mathbf{A}, f, n)$ is $\mathbf{A}[\Omega_1] = \mathbf{A}[\Omega_2]$ which is isomorphic to the polynomial algebra $\mathbf{A}[t]$.

Proof. From the defining relations of $S(\mathbf{A}, f, n)$, we have $[y, x] = yx - xy = f(e) = u(e + n) - u(e)$ and therefore $\Omega_2 = 2\Omega_1$.

Let us now prove that Ω_1 is central. As $\Omega_1 \in \mathbf{A}[xy, e] = S(\mathbf{A}, f, n)^0$ (Remark 5.1.5), we see that Ω_1 commutes with e .

From the defining relations of $S(\mathbf{A}, f, n)$ we have also $[e, x] = ex - xe = nx$, hence $ex = x(e + n)$ and therefore, for any $k \in \mathbb{N}$, $e^k x = x(e + n)^k$.

This implies of course that for any polynomial $P \in \mathbf{A}[t]$ we have

$$P(e)x = xP(e + n) \text{ or } P(e - n)x = xP(e). \quad (5-1-2)$$

Similarly one proves that

$$P(e)y = yP(e - n) \text{ or } P(e + n)y = yP(e). \quad (5-1-3)$$

Let us now show that Ω_1 commutes with x . Using Lemma 5.1.6 and (5-1-2) we obtain:

$$\begin{aligned} x\Omega_1 &= x(xy - u(e)) = x^2y - xu(e) = x(yx - f(e)) - xu(e) \\ &= x(yx - u(e + n) + u(e)) - xu(e) = xyx - xu(e + n) = xyx - u(e)x \\ &= \Omega_1 x. \end{aligned}$$

A similar calculation using (5-1-3) shows that Ω_1 commutes also with y . Hence Ω_1 belongs to the center of $S(\mathbf{A}, f, n)$.

Let now z be a central element of $S(\mathbf{A}, f, n)$. Then $z \in S(\mathbf{A}, f, n)^0$. We have $S(\mathbf{A}, f, n)^0 = \mathbf{A}[xy, e] = \mathbf{A}[\Omega_1, e]$, and hence z can be written as follows:

$$z = \sum c_i(e) \Omega_1^i \quad (\text{finite sum})$$

where $c_i(e) \in \mathbf{A}[e]$.

We have:

$$\begin{aligned} 0 &= [z, x] = [\sum c_i(e) \Omega_1^i, x] = \sum [c_i(e), x] \Omega_1^i \\ &= \sum (c_i(e)x - xc_i(e)) \Omega_1^i = \sum x(c_i(e+n) - c_i(e)) \Omega_1^i \quad (\text{using } (5-1-2)) \\ &= x(\sum (c_i(e+n) - c_i(e)) \Omega_1^i) \end{aligned}$$

As the algebra $S(\mathbf{A}, f, n)$ has no zero divisors we get:

$$\sum (c_i(e+n) - c_i(e)) \Omega_1^i = 0 \quad (*)$$

Let us now prove that the elements $e^j \Omega_1^i$ ($i, j \in \mathbb{N}$) are free over \mathbf{A} (this will prove that $\mathbf{A}[\Omega_1]$ is a polynomial algebra). Suppose that we have

$$\sum_{i,j} \alpha_{i,j} e^j \Omega_1^i = 0 \quad \text{with } \alpha_{i,j} \in \mathbf{A}.$$

As $\Omega_1 = xy - u(e)$, we have

$$\Omega_1^i = x^i y^i \text{ modulo monomials of the form } e^k x^p y^p \text{ with } p < i.$$

Therefore for all i, j , we have $\alpha_{i,j} = 0$. Then from (*) and from Corollary 5.1.4 above we obtain $c_i(e+n) - c_i(e) = 0$, for all i . As the elements e^k are free over \mathbf{A} (again Corollary 5.1.4) we obtain from Lemma 5.1.6 that $c_i \in \mathbf{A}$, for all i . □

Remark 5.1.8. Let $u \in \mathbf{A}[t]$. Define $f \in \mathbf{A}[t]$ by $f(t) = u(t+n) - u(t)$ (see Lemma 5.1.6). Then, from the definitions we have:

$$U(\mathbf{A}, u, n) = S(\mathbf{A}, f, n)/(xy - u(e)) = S(\mathbf{A}, f, n)/(\Omega_1)$$

where $(xy - u(e)) = (\Omega_1)$ is the ideal (automatically two-sided) generated by $xy - u(e) = \Omega_1$. Again, as for $S(\mathbf{A}, f, n)$, the adjoint action of \tilde{e} gives a decomposition of $U(\mathbf{A}, u, n)$ into weight spaces:

$$U(\mathbf{A}, u, n) = \oplus_{\nu \in \mathbb{Z}} U(\mathbf{A}, u, n)^\nu \quad (5-1-4)$$

where $U(\mathbf{A}, u, n)^\nu = \{\tilde{v} \in U(\mathbf{A}, u, n), [\tilde{e}, \tilde{v}] = \nu n \tilde{v}\}$.

Proposition 5.1.9.

Let $u \in \mathbf{A}[t]$ and $s \in \mathbb{N}$. The \mathbf{A} -linear mappings

$$\begin{array}{ccc} \varphi : \mathbf{A}[t] & \longrightarrow & U(\mathbf{A}, u, n) \\ P & \longmapsto & \varphi(P) = \tilde{x}^s P(\tilde{e}) \end{array} \quad \begin{array}{ccc} \psi : \mathbf{A}[t] & \longrightarrow & U(\mathbf{A}, u, n) \\ P & \longmapsto & \psi(P) = \tilde{y}^s P(\tilde{e}) \end{array}$$

are injective (in particular the subalgebra $\mathbf{A}[\tilde{e}] \subset U(\mathbf{A}, u, n)$ generated by \tilde{e} is a polynomial algebra).

Proof. Define $f(t) = u(t + n) - u(t)$. Every element of $S(\mathbf{A}, f, n)$ can be written uniquely in the form

$$\sum a_{k,\ell,m} e^k x^\ell y^m \quad (a_{k,\ell,m} \in \mathbf{A})$$

(Corollary 5.1.4). Therefore, from Remark 5.1.8, every element in $U(\mathbf{A}, u, n)$ can be written in the form:

$$\sum a_{k,\ell,m} \tilde{e}^k \tilde{x}^\ell \tilde{y}^m \quad (a_{k,\ell,m} \in \mathbf{A}).$$

Let $P(t) = \sum_{i=0}^p a_i t^i$, ($a_i \in \mathbf{A}$) be a polynomial such that $\tilde{x}^s P(\tilde{e}) = 0$ (i.e. $P \in \ker \varphi$). As $U(\mathbf{A}, u, n) = S(\mathbf{A}, f, n)/(\Omega_1)$, we see that there exists $\alpha \in S(\mathbf{A}, f, n)$ such that

$$x^s \sum_{i=0}^p a_i e^i = \alpha \Omega_1 = \alpha(xy - u(e)).$$

If $\alpha = \sum a_{k,\ell,m} e^k x^\ell y^m$, using the fact that $\Omega_1 = xy - u(e)$ is central and relation (5-1-2) we get:

$$\begin{aligned} x^s \sum_{i=0}^p a_i e^i &= \left(\sum_{k,\ell,m} a_{k,\ell,m} e^k x^\ell y^m \right) (xy - u(e)) = \sum_{k,\ell,m} a_{k,\ell,m} e^k x^\ell (xy - u(e)) y^m \\ &= \sum_{k,\ell,m} a_{k,\ell,m} e^k x^{\ell+1} y^{m+1} - \sum_{k,\ell,m} a_{k,\ell,m} e^k x^\ell u(e) y^m \\ &= \sum_{k,\ell,m} a_{k,\ell,m} e^k x^{\ell+1} y^{m+1} - \sum_{k,\ell,m} a_{k,\ell,m} e^k u(e - \ell n) x^\ell y^m \quad (**) \end{aligned}$$

Suppose now that $\alpha \neq 0$, then one can define

$$\ell_0 = \max\{\ell \in \mathbb{N}, \exists k, m, a_{k,\ell,m} \neq 0\}.$$

Let k_0, m_0 be such that $a_{k_0, \ell_0, m_0} \neq 0$. From (**) above we get:

$$x^s \sum_{i=0}^p a_i e^i + \sum_{k,\ell,m} a_{k,\ell,m} e^k u(e - \ell n) x^\ell y^m = \sum_{k,\ell,m} a_{k,\ell,m} e^k x^{\ell+1} y^{m+1}.$$

Using again (5-1-2) we obtain:

$$\sum_{i=0}^p a_i (e - ns)^i x^s + \sum_{k,\ell,m} a_{k,\ell,m} e^k u(e - \ell n) x^\ell y^m = \sum_{k,\ell,m} a_{k,\ell,m} e^k x^{\ell+1} y^{m+1}.$$

The left hand side of the preceding equality does not contain the monomial $e^{k_0} x^{\ell_0+1} y^{m_0+1}$, whereas the right hand side does. As the elements $e^k x^\ell y^m$ are a basis over \mathbf{A} (Corollary 5.1.4), we obtain a contradiction. Therefore $\alpha = 0$, and hence $x^s \sum_{i=0}^p a_i e^i = 0$, and again from Corollary 5.1.4, we obtain that $a_i = 0$ for all i . This proves that $\ker \varphi = \{0\}$. The proof for ψ is similar. \square

Corollary 5.1.10. *Every element \tilde{u} in $U(\mathbf{A}, u, n)$ can be written uniquely in the form*

$$\tilde{u} = \sum_{\ell > 0, k \geq 0} \alpha_{k,\ell} \tilde{y}^\ell \tilde{e}^k + \sum_{m \geq 0, r \geq 0} \beta_{m,r} \tilde{x}^m \tilde{e}^r$$

with $\alpha_{k,\ell}, \beta_{m,r} \in \mathbf{A}$.

Proof. From Corollary 5.1.4 and Remark 5.1.8 we know that any element in $U(\mathbf{A}, u, n)$ can be written (in a non unique way) as a linear combination, with coefficients in \mathbf{A} , of the elements $\tilde{x}^i \tilde{y}^j \tilde{e}^k$.

Suppose that $i \geq j$. Then we have:

$$\tilde{x}^i \tilde{y}^j \tilde{e}^k = \tilde{x}^{i-j} \tilde{x}^j \tilde{y}^j \tilde{e}^k.$$

As $\tilde{y}\tilde{x} = u(\tilde{e} + n)$ and $\tilde{x}\tilde{y} = u(\tilde{e})$, we see that $\tilde{x}^j \tilde{y}^j = Q_j(\tilde{e})$, where Q_j is a polynomial with coefficients in \mathbf{A} . Therefore $\tilde{x}^i \tilde{y}^j \tilde{e}^k = \sum_{\ell} \gamma_{\ell} \tilde{x}^{i-j} \tilde{e}^{\ell}$, with $\gamma_{\ell} \in \mathbf{A}$. Similarly one can prove that if $i < j$, we have $\tilde{x}^i \tilde{y}^j \tilde{e}^k = \sum_{\ell} \delta_{\ell} \tilde{y}^{j-i} \tilde{e}^{\ell}$, with $\delta_{\ell} \in \mathbf{A}$. This shows that any element \tilde{u} in $U(\mathbf{A}, u, n)$ can be written in the expected form.

Suppose now that:

$$\sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} \tilde{y}^{\ell} \tilde{e}^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} \tilde{x}^m \tilde{e}^r = 0.$$

Then, as $\tilde{y}^{\ell} \tilde{e}^k \in U(\mathbf{A}, u, n)^{-\ell}$ and $\tilde{x}^m \tilde{e}^r \in U(\mathbf{A}, u, n)^m$, we deduce from (5-1-4) that

$$\forall \ell > 0, \sum_k \alpha_{k, \ell} \tilde{y}^{\ell} \tilde{e}^k = 0, \quad \forall m \geq 0, \sum_r \beta_{m, r} \tilde{x}^m \tilde{e}^r = 0.$$

Then from Proposition 5.1.9, we deduce that $\alpha_{k, \ell} = 0$ and $\beta_{m, r} = 0$. □

5.2. Generators and relations for $D(V)^{G'}$.

Let $\mathcal{Z}(\mathcal{T})[t]$ be the polynomials in one variable with coefficients in $\mathcal{Z}(\mathcal{T})$. From the commutation rules $[E, X] = EX - XE = d_0 X$ and $[E, Y] = -d_0 Y$, we easily deduce that for $P \in \mathcal{Z}(\mathcal{T})[t]$ we have

$$YP(E) = P(E + d_0)Y, \quad XP(E) = P(E - d_0)X. \quad (5-2-1)$$

From Proposition 3.3.6 above, we know that any element in $D(V)^G$ can be written uniquely as a polynomial in E with coefficients in $\mathcal{Z}(\mathcal{T})$. As XY and YX belong to $D(V)^G$, there exist therefore two uniquely determined polynomials u_{XY} and $u_{YX} \in \mathcal{Z}(\mathcal{T})[t]$ such that $XY = u_{XY}(E)$ and $YX = u_{YX}(E)$. From (5-2-1) we obtain that

$$YXY = u_{YX}(E)Y = Yu_{XY}(E) = u_{XY}(E + d_0)Y$$

and therefore

$$u_{YX}(E) = u_{XY}(E + d_0) \quad (5-2-2)$$

As the polynomial u_{XY} will play an important role in Theorem 5.2.2 below, let us emphasize the connection between u_{XY} and the Bernstein polynomial b_Y . Remark first that $b_Y = b_{XY}$. We know from Corollary 3.3.8 that

$$\begin{aligned} h(XY)(\lambda + \rho) &= b_{XY}(\lambda) = b_Y(\lambda) \\ &= \sum_{i=0}^p \beta_i(\lambda + \rho)(a_0 d_0 + a_1 d_1 + \cdots + a_r d_r)^i = \sum_{i=0}^p \beta_i(\lambda + \rho)(h(E)(\lambda + \rho))^i \end{aligned}$$

with uniquely defined polynomials $\beta_i \in \mathbb{C}[A]^{W_0, \tau}$. Therefore we obtain:

Proposition 5.2.1.

Keeping the notations above, we have

$$u_{XY}(t) = \sum_{i=0}^p h^{-1}(\beta_i) t^i$$

Theorem 5.2.2. Let $f_{XY}(t) = u_{XY}(t + d_0) - u_{XY}(t)$. The mapping

$$\tilde{x} \mapsto X, \quad \tilde{y} \mapsto Y, \quad \tilde{e} \mapsto E$$

extends uniquely to an isomorphism of $\mathcal{Z}(\mathcal{T})$ -algebras between $U(\mathcal{Z}(\mathcal{T}), u_{XY}, d_0) \simeq S(\mathcal{Z}(\mathcal{T}), f_{XY}, d_0)/(\Omega_1)$ and $D(V)^{G'} = \mathcal{T}_0[X, Y]$.

Proof. As $[E, X] = d_0 X$, $[E, Y] = -d_0 Y$, $XY = u_{XY}(E)$ and $YX = u_{XY}(E + d_0)$ (see (5 - 2 - 2)), and as from Proposition 3.3.6 the algebra $D(V)^{G'} = \mathcal{T}_0[X, Y]$ is generated over $\mathcal{Z}(\mathcal{T})$ by X, Y, E , we know (universal property) that the mapping

$$\tilde{x} \mapsto X, \quad \tilde{y} \mapsto Y, \quad \tilde{e} \mapsto E$$

extends uniquely to a surjective morphism of $\mathcal{Z}(\mathcal{T})$ -algebras:

$$\varphi : U(\mathcal{Z}(\mathcal{T}), u_{XY}, d_0) \longrightarrow \mathcal{T}_0[X, Y].$$

From Corollary 5.1.10 any element \tilde{u} in $U(\mathcal{Z}(\mathcal{T}), u_{XY}, d_0)$ can be written uniquely in the form

$$\tilde{u} = \sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} \tilde{y}^\ell \tilde{e}^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} \tilde{x}^m \tilde{e}^r$$

with $\alpha_{k, \ell}, \beta_{m, r} \in \mathcal{Z}(\mathcal{T})$. Suppose now that $\tilde{u} \in \ker(\varphi)$, then

$$\varphi(\tilde{u}) = \sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} Y^\ell E^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} X^m E^r = 0,$$

with $\alpha_{k, \ell}, \beta_{m, r} \in \mathcal{Z}(\mathcal{T})$. then Corollary 3.3.7 implies that $\alpha_{k, \ell} = \beta_{m, r} = 0$. Hence φ is an isomorphism. \square

6. RADIAL COMPONENTS

6.1. Radial components and Bernstein-Sato polynomials.

For $(a_1, a_2, \dots, a_r) \in \mathbb{N}^r$ we define $\tilde{\mathbf{a}} = (0, a_1, a_2, \dots, a_r)$. By abuse of notation we will also write $(w, \tilde{\mathbf{a}}) = (w, a_1, a_2, \dots, a_r)$ for any variable w , and we sometimes consider $\tilde{\mathbf{a}}$ as an element of \mathbb{N}^r . We know from (2 - 2 - 4) that if $\mathbf{a} = (a_0, a_1, \dots, a_r)$, then $V_{\mathbf{a}} = \Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$. We know also from Proposition 2.2.6 that the spaces $U_{\tilde{\mathbf{a}}} = \bigoplus_{a_0 \in \mathbb{N}} \Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$ are the G' -isotypic components of $\mathbb{C}[V]$ and that the spaces $W_{\tilde{\mathbf{a}}} = \bigoplus_{a_0 \in \mathbb{Z}} \Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$ are the G' -isotypic components of $\mathbb{C}[\Omega]$. Therefore the algebra $D(V)^{G'} = \mathcal{T}_0[X, Y]$ stabilizes each space $U_{\tilde{\mathbf{a}}}$ and the algebra $D(\Omega)^{G'} = \mathcal{T}_0[X, X^{-1}] = \mathcal{T}$ stabilizes each space $W_{\tilde{\mathbf{a}}}$.

Let us consider the restriction map:

$$\begin{aligned} D(\Omega)^{G'} &\longrightarrow \text{End}(W_{\tilde{\mathbf{a}}}) \\ D &\longmapsto r_{\tilde{\mathbf{a}}}(D) = D|_{W_{\tilde{\mathbf{a}}}} \end{aligned}$$

Definition 6.1.1. Let $D \in D(\Omega)^{G'} = \mathcal{T}_0[X, X^{-1}] = \mathcal{T}$. The operator $r_{\tilde{\mathbf{a}}}(D) = D|_{W_{\tilde{\mathbf{a}}}}$ is called the radial component of D with respect to $\tilde{\mathbf{a}}$.

Example 6.1.2. Consider the case where $\tilde{\mathbf{a}} = 0$. Then $W_{\tilde{\mathbf{a}}} = \mathbb{C}[\Delta_0, \Delta_0^{-1}]$, and $r_0(D) = \overline{D}$ is the endomorphism of $\mathbb{C}[t, t^{-1}]$ defined by $D(\varphi \circ \Delta_0) = \overline{D}(\varphi) \circ \Delta_0$. The operator \overline{D} is the usual radial component of D (we will see below that \overline{D} is a differential operator).

Notice now that the space $W_{\tilde{\mathbf{a}}} = \oplus_{a_0 \in \mathbb{N}} \Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$ can be viewed as the space of Laurent polynomials in Δ_0 , with coefficients in $V_{\tilde{\mathbf{a}}}$, in other words any $P \in W_{\tilde{\mathbf{a}}}$ can be written uniquely under the form

$$P = \sum \Delta_0^p \gamma_p$$

with $\gamma_p \in V_{\tilde{\mathbf{a}}}$. This can also be written as $P = \varphi \circ (\Delta_0)$, with $\varphi(t) = \sum t^p \gamma_p \in V_{\tilde{\mathbf{a}}}[t, t^{-1}]$ (where $V_{\tilde{\mathbf{a}}}[t, t^{-1}]$ is precisely the set of linear combinations $\sum t^p \gamma_p$, with $\gamma_p \in V_{\tilde{\mathbf{a}}}$).

There is a natural action of $D(\mathbb{C}^*) = \mathbb{C}[t, t^{-1}, t \frac{d}{dt}]$ on $V_{\tilde{\mathbf{a}}}[t, t^{-1}]$ given by $\frac{d}{dt} t^p \gamma_p = p t^{p-1} \gamma_p$.

Proposition 6.1.3.

Let $D \in \mathcal{T}_n$ a homogeneous element of degree n . Let b_D be its Bernstein-Sato polynomial. Let $\varphi \in V_{\tilde{\mathbf{a}}}[t, t^{-1}]$. Then $D(\varphi \circ \Delta_0) = (t^n b_D(t \frac{d}{dt}, a_1, \dots, a_r) \varphi) \circ \Delta_0$, in other words $r_{\tilde{\mathbf{a}}}(D) = t^n b_D(t \frac{d}{dt}, a_1, \dots, a_r)$.

Proof. It is enough to show that the two operators coincide on elements of the form $\Delta_0^p \gamma_p$, with $\gamma_p \in V_{\tilde{\mathbf{a}}}$. Then $\varphi = t^p \gamma_p$. Let us write

$$b_D(\mathbf{a}) = \sum_k c_k(a_1, \dots, a_r) a_0^k$$

We have:

$$\begin{aligned} (t^n b_D(t \frac{d}{dt}, a_1, \dots, a_r) \varphi) \circ \Delta_0 &= t^n (\sum_k c_k(a_1, \dots, a_r) (t \frac{t}{dt})^k \varphi) \circ \Delta_0 \\ &= t^n (\sum_k c_k(a_1, \dots, a_r) p^k t^p \gamma_p) \circ \Delta_0 \\ &= (t^n b_D(p, a_1, \dots, a_r) t^p \gamma_p) \circ \Delta_0 = b_D(p, a_1, \dots, a_r) \Delta_0^{p+n} \gamma_p \\ &= D(\Delta_0^p \gamma_p) \end{aligned}$$

□

Remark 6.1.4. From Theorem 4.2.3 we remark that $r_{\tilde{\mathbf{a}}}(D)$ is nothing else but $\Psi^{-1}(D)(t, t^{-1}, t \frac{d}{dt}, a_1, \dots, a_r)$.

Corollary 6.1.5.

If (G, V) is a PV of commutative parabolic of rank $r + 1$, then the radial component of Y is given by

$$r_{\tilde{\mathbf{a}}}(Y) = t^{-1} \prod_{j=0}^r (t \frac{d}{dt} + a_1 + \dots + a_j + j \frac{d}{2})$$

Proof. This is just a consequence of the formula for b_Y given in Example 3.1.2

□

Example 6.1.6. Consider the case $A_{2(n-1)+1}$ in Table 1. In this case $G = S(GL(n) \times GL(n)) = \{(g_1, g_2) \in GL(n) \times GL(n) \mid \det(g_1 g_2) = 1\}$, $V = M_n$ is the full matrix space of size n , and the action is given by $(g_1, g_2).X = g_1 X g_2^{-1}$, for $X \in V$. Then $\Delta_0 = \det$ and

$$Y = \Delta_0(\partial) = \det\left(\frac{d}{dx_{ij}}\right)$$

where x_{ij} are the coefficients of the matrix X . As in this case $\frac{d}{2} = 1$ (see [M-R-S], table 2 p. 122), we have $b_Y(a_0, a_1, \dots, a_{n-1}) = \prod_{j=0}^{n-1} (a_0 + a_1 + \dots + a_j + j)$. Therefore the ordinary radial component $\bar{Y} = r_0(Y)$ defined by $\det(\frac{d}{dx_{ij}})(\varphi \circ \det) = (\bar{Y}\varphi) \circ \det$ is given by

$$\bar{Y} = t^{-1} \prod_{j=0}^{n-1} \left(t \frac{d}{dt} + j\right).$$

This radial component has already been calculated by Raïs ([Ra], p.22), by other methods. He obtained that $\bar{Y} = [\prod_{j=2}^{n-1} (t \frac{d}{dt} + j)] \frac{d}{dt}$. A simple calculation shows that the two operators are the same.

6.2. Algebras of radial components.

Definition 6.2.1. The radial component algebra $R_{\tilde{\mathbf{a}}}$ is the image of $D(V)^{G'} = \mathcal{T}_0[X, Y]$ under the map $D \mapsto r_{\tilde{\mathbf{a}}}(D)$.

Remember from Proposition 3.3.3 that the elements D in $\mathcal{Z}(\mathcal{T})$ are characterized by the fact that the corresponding Bernstein-Sato polynomial b_D does not depend on the a_0 variable. Therefore such a D acts by the scalar $b_D(\tilde{\mathbf{a}})$ on $W_{\tilde{\mathbf{a}}}$, that is $r_{\tilde{\mathbf{a}}}(D) = b_D(\tilde{\mathbf{a}})I_{|W_{\tilde{\mathbf{a}}}}$.

Let us consider the polynomial $u_{XY} \in \mathcal{Z}(\mathcal{T})[t]$ which was introduced in section 5.2. If $u_{XY} = \sum_j c_i t^i$, with $c_i \in \mathcal{Z}(\mathcal{T})$, we define

$$r_{\tilde{\mathbf{a}}}(u_{XY}) = \sum_j r_{\tilde{\mathbf{a}}}(c_i) t^i \in \mathbb{C}[t].$$

Theorem 6.2.2.

The radial component algebra $R_{\tilde{\mathbf{a}}}$ is isomorphic, as an associative algebra over \mathbb{C} , to the algebra $U(\mathbb{C}, r_{\tilde{\mathbf{a}}}(u_{XY}), d_0)$ introduced in Definition 5.1.1.

Proof. The algebra $R_{\tilde{\mathbf{a}}}$ is generated over \mathbb{C} by the elements $r_{\tilde{\mathbf{a}}}(E), r_{\tilde{\mathbf{a}}}(X), r_{\tilde{\mathbf{a}}}(Y)$. The defining relations of $U(\mathbb{C}, r_{\tilde{\mathbf{a}}}(u_{XY}), d_0)$ are verified:

- $[r_{\tilde{\mathbf{a}}}(E), r_{\tilde{\mathbf{a}}}(X)] = r_{\tilde{\mathbf{a}}}([E, X]) = d_0 r_{\tilde{\mathbf{a}}}(X)$
- $[r_{\tilde{\mathbf{a}}}(E), r_{\tilde{\mathbf{a}}}(Y)] = r_{\tilde{\mathbf{a}}}([E, Y]) = -d_0 r_{\tilde{\mathbf{a}}}(Y)$
- $r_{\tilde{\mathbf{a}}}(X) r_{\tilde{\mathbf{a}}}(Y) = r_{\tilde{\mathbf{a}}}(XY) = r_{\tilde{\mathbf{a}}}(u_{XY})(r_{\tilde{\mathbf{a}}}(E))$
- $r_{\tilde{\mathbf{a}}}(Y) r_{\tilde{\mathbf{a}}}(X) = r_{\tilde{\mathbf{a}}}(YX) = r_{\tilde{\mathbf{a}}}(u_{XY})(r_{\tilde{\mathbf{a}}}(E) + d_0)$.

Therefore the mapping

$$\tilde{x} \mapsto r_{\tilde{\mathbf{a}}}(X), \quad \tilde{y} \mapsto r_{\tilde{\mathbf{a}}}(Y), \quad \tilde{e} \mapsto r_{\tilde{\mathbf{a}}}(E)$$

extends uniquely to a surjective morphism of \mathbb{C} -algebras

$$\varphi_{\tilde{\mathbf{a}}} : U(\mathbb{C}, r_{\tilde{\mathbf{a}}}(u_{XY}), d_0) \longrightarrow R_{\tilde{\mathbf{a}}}.$$

From Corollary 5.1.10 any element \tilde{u} in $U(\mathbb{C}, r_{\tilde{\mathbf{a}}}(u_{XY}), d_0)$ can be written uniquely in the form

$$\tilde{u} = \sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} \tilde{y}^\ell \tilde{e}^k + \sum_{m \geq 0, s \geq 0} \beta_{m, s} \tilde{x}^m \tilde{e}^s$$

with $\alpha_{k, \ell}, \beta_{m, s} \in \mathbb{C}$. Suppose now that $\tilde{u} \in \ker(\varphi_{\tilde{\mathbf{a}}})$, then

$$\varphi_{\tilde{\mathbf{a}}}(\tilde{u}) = \sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} r_{\tilde{\mathbf{a}}}(Y)^\ell r_{\tilde{\mathbf{a}}}(E)^k + \sum_{m \geq 0, s \geq 0} \beta_{m, s} r_{\tilde{\mathbf{a}}}(X)^m r_{\tilde{\mathbf{a}}}(E)^s = 0.$$

Applying this operator to a function of the form $\Delta^{a_0} P$, with $P \in V_{\tilde{\mathbf{a}}}$, we obtain:

$$\sum_{\ell > 0} Y^\ell \left(\sum_{k \geq 0} \alpha_{k, \ell} E^k \Delta^{a_0} P \right) + \sum_{m \geq 0} X^m \left(\sum_{s \geq 0} \beta_{m, s} E^s \Delta^{a_0} P \right) = 0.$$

As the operators X and Y have degree d_0 and $-d_0$ respectively, this implies that

$$\forall \ell, \quad \sum_{k \geq 0} \alpha_{k, \ell} E^k \Delta^{a_0} P = 0, \quad \text{and} \quad \forall m, \quad \sum_{s \geq 0} \beta_{m, s} E^s \Delta^{a_0} P = 0.$$

As $E \Delta^{a_0} P = (a_0 d_0 + d(\tilde{\mathbf{a}})) \Delta^{a_0} P$, where $d(\tilde{\mathbf{a}}) = a_1 d_1 + \dots + a_r d_r$, we obtain:

$$\forall \ell, \quad \sum_{k \geq 0} \alpha_{k, \ell} (a_0 d_0 + d(\tilde{\mathbf{a}}))^k \Delta^{a_0} P = 0, \quad \text{and}$$

$$\forall m, \quad \sum_{s \geq 0} \beta_{m, s} (a_0 d_0 + d(\tilde{\mathbf{a}}))^s \Delta^{a_0} P = 0.$$

Hence

$$\forall \ell, \quad \sum_{k \geq 0} \alpha_{k, \ell} (a_0 d_0 + d(\tilde{\mathbf{a}}))^k = 0, \quad \text{and} \quad \forall m, \quad \sum_{s \geq 0} \beta_{m, s} (a_0 d_0 + d(\tilde{\mathbf{a}}))^s = 0.$$

As a_0 is an arbitrary element of \mathbb{Z} , this implies that $\forall(\ell, k)$ and $\forall(m, s)$, we have $\alpha_{k, \ell} = 0$ and $\beta_{m, s} = 0$. Hence $\tilde{u} = 0$ and $\varphi_{\tilde{\mathbf{a}}}$ is injective. \square

Remark 6.2.3. For $\tilde{\mathbf{a}} = 0$, the preceding result was first obtained by T. Levasseur ([Lev]), by other methods.

Define now $J_{\tilde{\mathbf{a}}} = \ker(r_{\tilde{\mathbf{a}}}|_{D(V)^{G'}})$. $J_{\tilde{\mathbf{a}}}$ is a two-sided ideal of $D(V)^{G'} = \mathcal{T}_0[X, Y]$.

Remember from Proposition 3.1.7 that any $D \in D(V)^{G'}$ can be written uniquely in the form:

$$D = \sum_{k \in \mathbb{N}^*} u_k Y^k + \sum_{n \in \mathbb{N}} v_n X^n \text{ (finite sum)}$$

where $u_k, v_n \in \mathcal{T}_0 = D(V)^G$.

Lemma 6.2.4.

$$J_{\tilde{\mathbf{a}}} = \{D = \sum_{k \in \mathbb{N}^*} u_k Y^k + \sum_{n \in \mathbb{N}} v_n X^n \mid u_k, v_n \in J_{\tilde{\mathbf{a}}} \cap \mathcal{T}_0\}.$$

Proof. From Theorem 6.2.2 the algebra $R_{\tilde{\mathbf{a}}}$ is isomorphic to $U(\mathbb{C}, r_{\tilde{\mathbf{a}}}(u_{XY}), d_0)$. If $r_{\tilde{\mathbf{a}}}(D) = \sum_{k \in \mathbb{N}^*} r_{\tilde{\mathbf{a}}}(u_k) r_{\tilde{\mathbf{a}}}(Y)^k + \sum_{n \in \mathbb{N}} r_{\tilde{\mathbf{a}}}(v_n) r_{\tilde{\mathbf{a}}}(X)^n = 0$, then, from Corollary 5.1.10, we obtain that $r_{\tilde{\mathbf{a}}}(u_k) = 0$ and $r_{\tilde{\mathbf{a}}}(v_n) = 0$ for all k and all n . \square

Remember that d_0 denotes the degree Δ_0 and that $d(\tilde{\mathbf{a}})$ denotes the degree of the elements in $V_{\tilde{\mathbf{a}}}$ (see section 2.2). Let us now give a set of generators for the ideal $\ker(r_{\tilde{\mathbf{a}}})$ in $D(V)^{G'} = \mathcal{T}_0[X, Y]$. From Proposition 6.1.3 we obtain that $r_{\tilde{\mathbf{a}}}(E) = d_0(t \frac{d}{dt}) + d(\tilde{\mathbf{a}})$. Therefore $r_{\tilde{\mathbf{a}}}(\frac{E - d(\tilde{\mathbf{a}})}{d_0}) = t \frac{d}{dt}$.

Define $G_i^{\tilde{\mathbf{a}}} = R_i - b_{R_i}((\frac{E - d(\tilde{\mathbf{a}})}{d_0}), \tilde{\mathbf{a}})$ where the R_i 's are the Capelli operators introduced in section 2.2. Using Proposition 6.1.3 again we obtain

$$\begin{aligned} r_{\tilde{\mathbf{a}}}(G_i^{\tilde{\mathbf{a}}}) &= r_{\tilde{\mathbf{a}}}(R_i - b_{R_i}((\frac{E - d(\tilde{\mathbf{a}})}{d_0}), \tilde{\mathbf{a}})) = r_{\tilde{\mathbf{a}}}(R_i) - b_{R_i}(r_{\tilde{\mathbf{a}}}(\frac{E - d(\tilde{\mathbf{a}})}{d_0}), \tilde{\mathbf{a}}) \\ &= b_{R_i}(t \frac{d}{dt}, \tilde{\mathbf{a}}) - b_{R_i}(t \frac{d}{dt}, \tilde{\mathbf{a}}) = 0. \end{aligned}$$

Hence the elements $G_i^{\tilde{\mathbf{a}}}$ belong to $J_{\tilde{\mathbf{a}}}$.

Theorem 6.2.5.

The elements $G_i^{\tilde{\mathbf{a}}}$ are generators of $J_{\tilde{\mathbf{a}}}$:

$$J_{\tilde{\mathbf{a}}} = \ker(r_{\tilde{\mathbf{a}}}|_{D(V)^{G'}}) = \sum_{i=0}^r D(V)^{G'} G_i^{\tilde{\mathbf{a}}} = \sum_{i=0}^r G_i^{\tilde{\mathbf{a}}} D(V)^{G'}.$$

Proof. From Lemma 6.2.4, it is now enough to prove that

$$J_{\tilde{\mathbf{a}}} \cap \mathcal{T}_0 \subset \sum_{i=0}^r D(V)^G G_i^{\tilde{\mathbf{a}}} = \sum_{i=0}^r \mathcal{T}_0 G_i^{\tilde{\mathbf{a}}}.$$

Let $D \in J_{\tilde{\mathbf{a}}} \cap \mathcal{T}_0$. As $\mathcal{T}_0 = \mathbb{C}[R_0, \dots, R_r]$ (Proposition 2.2.7), we have also $\mathcal{T}_0 = \mathbb{C}[G_0^{\tilde{\mathbf{a}}}, \dots, G_r^{\tilde{\mathbf{a}}}, E]$.

Therefore $D = \sum Q_i E^i$, where $Q_i \in \mathbb{C}[G_0^{\tilde{\mathbf{a}}}, \dots, G_r^{\tilde{\mathbf{a}}}]$. Hence $Q_i \in Q_i(0) + \sum_{i=0}^r D(V)^G G_i^{\tilde{\mathbf{a}}}$. Then

$$0 = r_{\tilde{\mathbf{a}}}(D) = \sum_i Q_i(0) r_{\tilde{\mathbf{a}}}(E^i) = \sum_i Q_i(0) (d_0(t \frac{d}{dt}) + d(\tilde{\mathbf{a}}))^i.$$

Therefore $Q_i(0) = 0$ ($i = 0, \dots, r$). Hence $Q_i \in \sum_{i=0}^r D(V)^G G_i^{\tilde{\mathbf{a}}}$, which yields $D \in \sum_{i=0}^r D(V)^G G_i^{\tilde{\mathbf{a}}}$. \square

Remark 6.2.6. For $\tilde{\mathbf{a}} = 0$, the result of the preceding Theorem is due to T. Levasseur ([Lev], Theorem 4.11. (v)).

6.3. Rational radial component algebras.

Definition 6.3.1. *The rational radial component algebra $R_{\tilde{\mathbf{a}}}^r$ is the image of $D(\Omega)^{G'} = \mathcal{T}_0[X, X^{-1}] = \mathcal{T}$ under the map $D \mapsto r_{\tilde{\mathbf{a}}}(D)$.*

In fact as shown in the following proposition the structure of the algebras $R_{\tilde{\mathbf{a}}}^r$ is more simpler than the structure of $R_{\tilde{\mathbf{a}}}$, and the ideal $I_{\tilde{\mathbf{a}}} = \ker(r_{\tilde{\mathbf{a}}}) \subset \mathcal{T}$ has the same generators as $J_{\tilde{\mathbf{a}}}$.

Proposition 6.3.2.

- 1) For all $\tilde{\mathbf{a}}$, the rational radial component algebra $R_{\tilde{\mathbf{a}}}^r$ is isomorphic to $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}]$.
- 2) $I_{\tilde{\mathbf{a}}} = \ker(r_{\tilde{\mathbf{a}}}) = \sum_{i=0}^r \mathcal{T}G_i^{\tilde{\mathbf{a}}} = \sum_{i=0}^r G_i^{\tilde{\mathbf{a}}} \mathcal{T}$.

Proof. 1) We have $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$. And $\mathcal{T}_0 = \mathcal{Z}(\mathcal{T})[E]$, from Proposition 3.3.6. Therefore $\mathcal{T} = \mathcal{Z}(\mathcal{T})[X, X^{-1}, E]$. On the other hand we have $r_{\tilde{\mathbf{a}}}(\mathcal{Z}(\mathcal{T})) = \mathbb{C}$, $r_{\tilde{\mathbf{a}}}(X) = t$, $r_{\tilde{\mathbf{a}}}(X^{-1}) = t^{-1}$ and $r_{\tilde{\mathbf{a}}}(E) = d_0(t \frac{d}{dt}) + d(\tilde{\mathbf{a}})$. Hence $R_{\tilde{\mathbf{a}}}^r = r_{\tilde{\mathbf{a}}}(\mathcal{T}) = \mathbb{C}[t, t^{-1}, d_0(t \frac{d}{dt}) + d(\tilde{\mathbf{a}})] = \mathbb{C}[t, t^{-1}, t \frac{d}{dt}]$.

2) Obviously $\sum_{i=0}^r \mathcal{T}G_i^{\tilde{\mathbf{a}}} \subset I_{\tilde{\mathbf{a}}}$. As $I_{\tilde{\mathbf{a}}}$ is a two-sided ideal, it is graded by Theorem 3.5.1. If $D \in I_{\tilde{\mathbf{a}}} \cap \mathcal{T}_p$, then $X^{-p}D \in \mathcal{T}_0 \cap I_{\tilde{\mathbf{a}}} = \mathcal{T}_0 \cap J_{\tilde{\mathbf{a}}} = \sum_{i=0}^r \mathcal{T}_0 G_i^{\tilde{\mathbf{a}}}$. Therefore $D \in \sum_{i=0}^r \mathcal{T}G_i^{\tilde{\mathbf{a}}}$. □

**Tables of indecomposable, saturated, multiplicity free representations
with one dimensional quotient**

Table 2: Irreducible representations
(Notations for representations as in [Be-Ra-2])

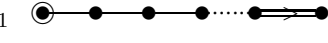
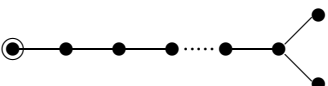
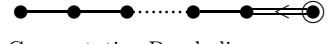
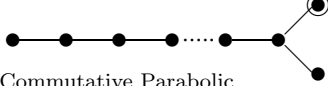
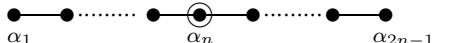

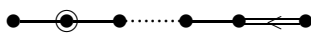
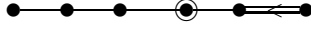
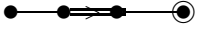

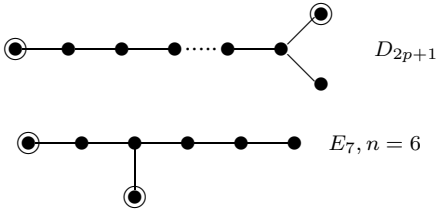
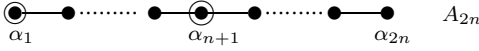
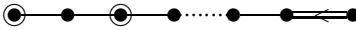
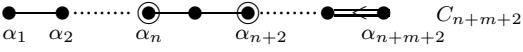
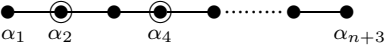
Representation, rank	Weighted Dynkin diagram (if parabolic type)	Regular	Fundamental invariant
$\mathbb{C}^* \times SO(n)$ ($n \geq 3$) rank=2	$n = 2p + 1$  B_{p+1} $n = 2p$  D_{p+1} Commutative Parabolic (both)	Yes	Non degenerate quadratic form
$\mathbb{C}^* \times S^2(SL(n))$ ($n \geq 2$) rank= n	 C_n Commutative Parabolic	Yes	Determinant on symmetric matrices
$\mathbb{C}^* \times \Lambda^2(SL(n))$ ($n \geq 4$) and $n = 2p$ rank=p	 D_{2p} Commutative Parabolic	Yes	Pfaffian on skew symmetric matrices
$\mathbb{C}^* \times (SL(n)^* \otimes SL(n))$ ($n \geq 2$) rank=n	 A_{2p-1} Commutative Parabolic	Yes	Determinant on full matrix space
$\mathbb{C}^* \times E_6$ (dim=27) rank=3	 E_7 Commutative Parabolic	Yes	Freudenthal cubic
$\mathbb{C}^* \times (SL(2) \otimes Sp(2n))$ ($n \geq 2$) rank=3	 C_{n+2}	Yes	$Pf(^tXJX)$ $X \in M(2n, 2)$ $Pf = Pfaffian$
$\mathbb{C}^* \times SL(4) \times Sp(4)$ rank=6	 C_6	Yes	Det(X), $X \in M(4)$
$\mathbb{C}^* \times Spin(7)$ rank=2	 F_4	Yes	Non degenerate quadratic form ($Spin(7) \hookrightarrow SO(8)$)
$\mathbb{C}^* \times Spin(9)$ rank=3	Non parabolic	Yes	Non degenerate quadratic form
$\mathbb{C}^* \times G_2$ (dim = 7) rank=2	Non parabolic	Yes	Non degenerate quadratic form $G_2 \hookrightarrow SO(7)$

Table 3: Non Irreducible representations
(Notations for representations as in [Be-Ra-2])

Representation	Weighted Dynkin diagram (if parabolic type)	Regular	Fundamental invariant
$(\mathbb{C}^*)^2 \times (SL(n)^* \oplus_{SL(n)} SL_n)$ $n \geq 2$ rank=3	 A_{n+1}	Yes	$f(u, v) = uv$ on $M(1, n) \oplus M(n, 1)$
(a) $(\mathbb{C}^*)^2 \times (SL(n) \oplus_{SL(n)} \Lambda^2(SL(n)))$ $(n \geq 4, n = 2p \text{ even})$ rank=n=2p (b) $(\mathbb{C}^*)^2 \times (SL(6)^* \oplus_{SL(6)} \Lambda^2(SL(6)))$ rank=6	 D_{2p+1} $E_7, n = 6$	No	Pfaffian on skew symmetric matrices (on 2nd component)
(a) $(\mathbb{C}^*)^2 \times SL(n) \oplus_{SL(n)} (SL(n) \otimes SL(n)), n \geq 2$ (b) $(\mathbb{C}^*)^2 \times SL(n)^* \oplus_{SL(n)} (SL(n) \otimes SL(n)), n \geq 3$ rank=2n	 A_{2n}	No	Determinant on full matrix space (on 2nd component)
$(\mathbb{C}^*)^2 \times SL(2) \oplus_{SL(2)} (SL(2) \otimes Sp(2n)), n \geq 2$ rank=5	 C_{n+3}	No	$Pf({}^tXJX)$ $X \in M(2n, 2)$ $Pf = Pfaffian$ (on 2nd component)
$(\mathbb{C}^*)^2 \times (SL(n) \otimes SL(2)) \oplus_{SL(2)} (SL(2) \otimes Sp(2m))$ $(n \geq 3, m \geq 2)$ rank=6	 C_{n+m+2}	No	$Pf({}^tXJX)$ $X \in M(2m, 2)$ $Pf = Pfaffian$ (on 2nd component)
$(\mathbb{C}^*)^2 \times (SL(2) \otimes SL(2)) \oplus_{SL(2)} (SL(2) \otimes SL(n))$ $(n \geq 3)$ rank=5	 A_{n+3}	No	$Det(X), X \in M(2, 2)$ (on 1st component)
$(\mathbb{C}^*)^2 \times (Sp(2n) \oplus_{Sp(2n)} Sp(2n)), n \geq 2$ rank=4	Non parabolic	Yes	$f(u, v) = {}^t uv$ on $M(1, 2n) \oplus M(1, 2n)$

REFERENCES

- [Be-Ra-1] C. BENSON, G. RATCLIFF — *On multiplicity free actions*, in "Representations of Real and p-adic groups", 221-304. Lecture Notes Series 2. Institute of Mathematical Sciences, National University of Singapore, Singapore University Press, World Scientific, 2004
- [Be-Ra-2] C. BENSON, G. RATCLIFF — *A Classification of Multiplicity Free Actions*, J. Alg. 181, 152-186 (1996).
- [B-R] N. BOPP, H. RUBENTHALER — *Fonction zêta associée à la série principale sphérique de certains espaces symétriques*, Ann. Sci. Ecole Norm. Sup. 26 (1993) n°6, 701-745.
- [Bou-1] N. BOURBAKI — *Groupes et Algèbres de Lie, Chapitre 1*, C.C.L.S, Paris, 1971.
- [Br] M. BRION — *Représentations exceptionnelles des groupes semi-simples*, Ann. Sci. Ecole Norm. Sup. 18 (1985), 345-387.

- [F-K] J. FARAUT, A. KORANYI — *Analysis on Symmetric Cones*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994. xii+382 pp.
- [G-W] K. R. GOODEARL, R. B. WARFIELD, — *An Introduction to Noncommutative Noetherian Rings*, Second edition. London Mathematical Society Student Texts, 61. Cambridge University Press, Cambridge, 2004. xxiv+344 pp.
- [H-S] G. HECKMANN, H. SCHLICHTKRULL — *Harmonic Analysis and Special Functions on Symmetric Spaces*, Perspectives in Mathematics Vol. 16, Academic Press (1994).
- [H] R. HOWE, — *Transcending Classical Invariant Theory*, Journal of the Amer. math. Soc. Volume 2, Number 3 (1989), 535-552.
- [H-U] R. HOWE, T. UMEDA — *The Capelli identity, the double commutant theorem, and multiplicity-free actions*, Math. Ann. 290 (1991), n°3, 565-619.
- [Hu] J. E. HUMPHREYS — *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990. xii+204 pp.
- [I] J. IGUSA — *On Lie Algebras Generated by Two Differential Operators*, Manifolds and Lie Groups, Progr. Math. 14 (1981) Birkhäuser, Boston, Mass., 187-195.
- [Ka] V. KAC — *Some remarks on nilpotent orbits*, J. Alg. 64 (1980), 190-213.
- [Ki] T. KIMURA — *Introduction to Prehomogeneous Vector Spaces*, Transl. Math. Monogr., vol 215, American Mathematical Society, Providence, RI, 2003.
- [Kn-1] F. KNOP — *A Harish-Chandra homomorphism for reductive group actions*, Ann. of Math. (2) 140 (1994), no. 2, 253-288.
- [Kn-2] F. KNOP — *Some remarks on multiplicity free spaces*, Representation theories and algebraic geometry (Montreal, PQ, 1997), 301-317, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Kluwer Acad. Publ., Dordrecht, 1998.
- [Ko] M. KOECHER — *Imbedding of Jordan algebras into Lie algebras I*, Americ. J. Math., 89 (1967), 787-815.
- [K-S] B. KOSTANT, S. SAHI — *The Capelli identity, Tube Domains and the Generalized Laplace Transform*, Adv. Math. 87 (1991), 71-92.
- [Lea] A. S. LEAHY — *A Classification of Multiplicity Free Representations*, Journal of Lie Theory, volume 8 (1998) 367-391.
- [Lev] T. LEVASSEUR — *Radial components, prehomogeneous vector spaces, and rational Cherednik algebras*, Int. Math. Res. Not. IMRN 2009, no. 3, 462-511.
- [L-S] T. LEVASSEUR and J.T. STAFFORD — *Rings of differential operators on classical rings of invariants*, Memoirs of the AMS, volume 81, n° 412 (1989).
- [L-M-O] A. LEROY, J. MATCZUK and J. OKNINSKI — *On the Gelfand-Kirillov dimension of normal localizations and twisted polynomial rings*, in Perspectives in Ring Theory, F. van Oystaeyen and L. Le Bruyn, Editors, Kluwer Academic Publishers (1988), 205-214.
- [M] H. MAASS — *Siegel's Modular Forms and Dirichlets series*, Lecture Notes in Mathematics, Vol. 216. Springer-Verlag, Berlin-New York, 1971. v+328 pp.
- [MC-R] J.C. McCONNEL and J.C. ROBSON — *Non Commutative Noetherian Rings*, Wiley-interscience, Chichester, 1987.
- [M-R-S] I. MULLER, H. RUBENTHALER and G. SCHIFFMANN — *Structure des espaces préhomogènes associés à certaines algèbres de Lie graduées*, Math. Ann. 274 (1986), 95-123.

- [No] T. NOMURA, — *Algebraically independent generators of invariant differential operators on a symmetric cone*, J. reine angew. Math. 400 (1989), 122-133.
- [Ra] M. RAÏS — *Distributions homogènes sur des espaces de matrices*, Bull. Soc. Math. France, Mémoire 30 (1972)
- [Ra-S] S. RALLIS and G. SCHIFFMANN — *Weil Representation. I Intertwining distributions and discrete spectrum*, Mem. Amer. Math. Soc. 25 (1980) n°231.
- [Ru-1] H. RUBENTHALER — *Espaces vectoriel préhomogènes, sous-groupes paraboliques et \mathfrak{sl}_2 - triplets*, C.R. Acad. Sci. Paris Sér. A-B 290 (1980), n°3, A127-A129.
- [Ru-2] H. RUBENTHALER — *Espaces préhomogènes de type parabolique*, Thèse d'Etat, Université de Strasbourg (1982).
- [Ru-3] H. RUBENTHALER — *Algèbres de Lie et espaces préhomogènes*, in: Travaux en cours, Hermann, Paris, 1992.
- [Ru-4] H. RUBENTHALER — *Une dualité de type Howe en dimension infinie*, C. R. Acad. Sci. Paris Sr. I Math. 314 (1992), no. 6, 435-440.
- [Ru-5] H. RUBENTHALER — *Algebras of invariant differential operators on a class of multiplicity free spaces*, C. R. Math. Acad. Sci. Paris 347 (2009), no. 23-24, 1343-1346.
- [R-S-1] H. RUBENTHALER and G. SCHIFFMANN — *Opérateurs différentiels de Shimura et espaces préhomogènes*, Invent. Math. 90 (1989), 409-442.
- [R-S-2] H. RUBENTHALER and G. SCHIFFMANN — *SL_2 -triplet associé à un polynôme homogène*, J. reine angew. math. 408 (1990), 136-158.
- [Sch] G. SCHWARZ — *Finite-dimensional representations of invariant differential operators*, J. Alg. 258 (2002), 160-204.
- [Sh] D. SHALE — *Linear symmetries of free boson fields*, Trans. Amer. Math. Soc. 103 (1962), 149-167.
- [S-K] M. SATO – T. KIMURA — *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. 65 (1977), 1-155.
- [Sm] S. P. SMITH — *A class of algebras similar to the enveloping algebra of $\mathfrak{sl}(2)$* , Trans. Amer. Math. Soc. 322, 1 (1990), 285-314.
- [Ter] A. TERRAS — *Harmonic analysis on symmetric spaces and applications I, II*, Springer (1988).
- [Ti] J. TITS — *Une classe d'algèbres de Lie en relation avec les algèbres de Jordan*, Indag. Math. 24, (1962), 530-535.
- [VdBe] M. VAN DEN BERGH — *Some rings of differential operators for SL_2 -invariants are simple*, J. Pure App. Algebra 107 (1996) 309-335.
- [V-K] E. VINBERG and B. KIMELFELD — *Homogeneous domains on flag manifolds and spherical subsets of semi-simple Lie groups*, Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 12-19, 96.
- [Wa] N. WALLACH — *Polynomial Differential Operators Associated with Hermitian Symmetric Spaces*, Representation Theory of Lie Groups and Lie Algebras, World Sci. Publishing, River Edge, NJ (1992), 76-94.
- [We] A. WEIL — *Sur certains groupes d'opérateurs unitaires*, Acta Math. 111 (1964), 143-211.
- [Y] Z. YAN — *Invariant Differential Operators and Holomorphic Function Spaces*, J. Lie Theory 10 (2000) n°1, 1-31.
- [Z] J.J. ZHANG — *A note on GK dimension of skew polynomial extensions*, Proc. Amer. Math. Soc. 125 (1997), no. 2, 363-373

HUBERT RUBENTHALER, INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ DE STRASBOURG ET CNRS, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE, E-MAIL: rubenth@math.u-strasbg.fr